SCIENCE SERIES

Engineering Science Series

MECHANICS OF MATERIALS

ENGINEERING SCIENCE SERIES

EDITED BY

DUGALD C. JACKSON

PROFESSOR OF ELECTRICAL ENGINEERING, MASSACHUSETTS INSTITUTE OF TECHNOLOGY

EARLE R. HEDRICK

PROFESSOR OF MATHEMATICS, UNIVERSITY OF CALIFORNIA

ALTERNATING CURRENTS AND ALTERNATING CURRENT MACHINERY
By D. C. JACKSON and J. P. JACKSON

DESCRIPTIVE GEOMETRY
By ERVIN KENISON and HARRY C. BRADLEY

DESCRIPTIVE GEOMETRY
By GEORGE YOUNG, JR., and HUBERT E. BAXTER

ELECTRIC AND MAGNETIC MEASUREMENT
By CHARLES M. SMITH

ENGINES AND BOILERS
By THOMAS T. EYRE

RIVER AND HARBOR CONSTRUCTION
By CURTIS McD. TOWNSEND

ELECTRICAL VIBRATION INSTRUMENTS
By A. E. KENNELLY

ELEMENTS OF ELECTRICAL ENGINEERING
By GEORGE D. SHEPARDSON

ELECTRICAL ENGINEERING
By L. A. HAZELTINE

PRINCIPLES OF TELEPHONE TRANSMISSION
By M. P. WEINBACH

PRINCIPLES OF MACHINE DESIGN
By C. A. NORMAN

TELEPHONE COMMUNICATION SYSTEMS
By R. G. KLOEFFLER

A TREATISE ON HYDRAULICS
By H. J. HUGHES and A. T. SAFFORD

MECHANICS OF MATERIALS
By GEORGE YOUNG, JR., and HUBERT E. BAXTER

MECHANICS OF MATERIALS

BY

GEORGE YOUNG, JR.
PROFESSOR OF ARCHITECTURE
CORNELL UNIVERSITY

AND

HUBERT EUGENE BAXTER
ASSISTANT PROFESSOR OF ARCHITECTURE
CORNELL UNIVERSITY

New York
THE MACMILLAN COMPANY
1927

Set up and electrotyped
Published September, 1927

PRINTED IN THE UNITED STATES OF AMERICA BY
THE BERWICK & SMITH CO.

PREFACE

This book is intended to serve as an introduction to the study of the mechanical aspects of construction. Such a study must inevitably deal with both fact and theory. In conducting their classes in the theory of construction, the authors have felt that the student should acquire his facts as a natural accretion around a core of theory and that empirical and experimental methods should form as small a part as possible of his early training. Therefore, little of an empirical nature appears in this book. Likewise no emphasis has been placed on the study of the physical properties of materials, for this subject should be developed in a separate course which includes some actual contact with the materials themselves and with the methods used in making scientific determinations of their properties.

It has not seemed advisable to present very much in the way of tabulated data though a few tables are included in the Appendix. Many good handbooks are available and it has been assumed that this book will be supplemented by a reference book substantially equivalent to those published by the steel manufacturers.

For reasons more fully explained in § 84, no attempt has been made to select topics and problems from the very latest developments in structural practice. The attempt has rather been to select such examples as afford a good opportunity for the study of some general principle. For example the flitched beam (§ 208) is well nigh obsolete, but the principle involved in its design leads directly to the study of reinforced concrete. Again it is quite possible that, with the development of welding, riveted joints may become more or less rare, but as an illustration of certain principles well worthy of study the riveted joint will remain the more important form.

The student is assumed to have an elementary knowledge of physics and of mathematics, including the integral calculus. The attempt has been to present each idea as required by the nature of the subject itself rather than to fit the treatment to students who are insufficiently prepared, on the one hand, or on

the other, to make the whole field an exercising ground for the higher mathematics.

Since structural work makes severe demands on the imagination and the ability to "see" solutions, the visual appeal has been emphasized whenever possible. Also physical concepts have been given preference over mathematical processes, graphical methods being used quite freely. Nevertheless, in the treatment of such subjects as the deformations of beams and unsymmetrical bending, well known graphical or partly graphical methods have been omitted and attention has been confined to developing the fundamental concepts on which such methods are based.

In preparing the illustrations the attempt has been made to keep them in scale wherever possible. In the study of such a subject as deformation, some exaggeration is positively necessary; but in other cases, notably in shear and moment diagrams, the actual scales have been followed closely in order to cultivate the sense of proportion and a keenness of observation that comes from a careful training in this respect.

In Chapters XX to XXV several subjects have been touched upon in a most rudimentary fashion. Each subject might well be the basis of a whole book. The idea has been to rouse interest and excite curiosity rather than to offer solutions.

In presenting such a subject, material is necessarily gathered from many sources. Among the books most used for reference are: "The Mechanics of Engineering" by Professor I. P. Church, "Mechanics of Materials" by Professor Mansfield Merriman and "Applied Mechanics" by Professors C. E. Fuller and W. A. Johnson. Members of the various Faculties of the University have been generously helpful with suggestions, advice and encouragement. Special acknowledgment is due to Professor C. F. Craig of the Department of Mathematics for constructive criticism of the most helpful sort. The illustrations of materials tested to destruction were taken from test made in the laboratories of the College of Engineering under the direction of Professors E. N. Burrows, A. C. Davis, and H. H. Schofield. Other acknowledgments are made throughout the text.

George Young, Jr.
H. E. Baxter.

CONTENTS

MECHANICS OF MATERIALS

MECHANICS OF MATERIALS

CHAPTER I

INTRODUCTION

1. Purpose and Scope. The structural problem consists in selecting and combining material to serve some human end, at a minimum expenditure of natural resources and human effort. If the complete structure fails to serve its purpose or if, in building it, more labor or more material has been expended than was necessary, there has been an irreparable economic loss.

Thus the design of structures involves two problems: *what* to build, and *how* to build it. The first problem has to do with needs to be satisfied and conditions to be met; such as the size, shape, and general disposition of rooms in the case of a building, and grades, clearances, loads to be carried, and circulation in the case of a bridge. Such considerations fix the broad general conception of *what* the structure is to be. The second problem has to do with the means by which the desired ends may be achieved, the materials to be used, and the size, shape, and disposition of the individual members.

These two problems are necessarily coexistent and interdependent. The size and disposition of rooms frequently depend on economical column spacings. The number of piers in a bridge may depend on the relative cost of foundation and superstructure. Again the architectural treatment or even the fundamental elements of the plan may be fixed with reference to the limitations of materials which are easily available. Special materials and processes may have to be evolved to meet definite needs. Every structure represents more or less of a struggle to reconcile what is wanted with how it may be accomplished. Thus the designer cannot safely neglect the structural problem, nor should the engineer allow purely structural considerations to hamper the usefulness, suitability, or character of the structure.

The purpose of the following chapters is to lay the foundation for the study of only one of these problems. It will be assumed that definite results are to be accomplished, and an answer will be sought as to how best to fulfill the requirements. If a beam or column is to be designed, the load to be carried and the material to be used will be specified, and a suitable size and shape will be determined. However, while we shall be concerned solely with the problem of how to do something, the student should not forget its relation to and its dependence upon the other problem of what, in the broader sense, is to be accomplished.

2. Elements of Structural Engineering. The elements which enter into the problem of structural engineering are (1) the loads which a structure or any part of it is to carry, (2) the materials of which it is to be made, and (3) the size, shape, and disposition of the various members.

The determination of the loads will be taken up only in its most rudimentary form. In practice it is a very simple subject, but it depends mainly on experience, statistics, and local conditions. This book will deal more especially with the principles of statics, and with the mechanics of materials, leaving the other questions mentioned above to be taken up in a later course in structural design.

The selection of materials which will be appropriate and economical for use in a given structure involves a detailed knowledge of the physical characteristics and properties of each possible material as well as a broad general acquaintance with market conditions. This is a lifelong study, since the sources of supply, the processes of manufacture, and the comparative costs are constantly changing.

The study of the characteristics of materials has been conducted intensively in the past few decades. There is now a vast amount of literature, both standard and periodical. The testing of materials can fairly be classed as a separate science which is well established and is being carried forward in numerous laboratories.

While the sources of information regarding materials are plentiful, it is probably impossible for a satisfactory knowledge of materials to be acquired merely through reading. Some direct contact with materials, both in the laboratory and in the field, should form part of the training of every person intrusted with their selection. The subject of materials will be discussed in this book only in so far as it is necessary to give point to the problems discussed.

When the loading of a structure has been determined and the material to be used has been selected, it remains to fix the size, shape, and disposition of its parts. It is this problem that is to be investigated in the following chapters. The results of experimental engineering on the one hand and the principles of mechanics on the other are brought to bear on the problem through processes which are chiefly mathematical.

3. Methods of Structural Engineering. Principles may be combined, elaborated, or even evolved by purely mental processes. Facts are established through the senses, by observation and by experiment. The methods of structural engineering spring from the above considerations.

A. *The rational method* consists in obtaining results by purely mental processes, reasoning from established facts to their logical conclusions. This is what is done in each of the branches of mathematics, which is a wholly rational science.

B. *The experimental method* is the one by which most of the facts of science are established. It consists in the observation of actual happenings and the classification of the determined results. Thus the acceleration due to gravity can be established only by experiment; but once the law is thus established, it may be extended and applied to wholly new cases by the rational method.

C. *The empirical method* consists in the study of precedents, and in following traditions with only such slight and cautious departures as may be dictated by time and circumstance.

The solution of any important structural problem will involve, to some extent, each of these methods. Applied sciences have,

in general, grown from the empirical, through the experimental, to a rational stage. The great structures of antiquity were built with a very meagre scientific knowledge concerning the mechanical and structural principles involved. Therefore progress was slow and halting. The developments during the Middle Ages and the Renaissance were, in essence, slow and cautious experiments. It is only in comparatively recent times that rational methods have been made possible by the body of fact and experience accumulated in the past and by the development of science in general. The rapid progress and dependable conclusions of the present day are due solely to the application of the rational method to the problem in hand.

Where the necessary facts are well established, the rational method is far the best and most reliable. But when the facts are at all obscure or confused, it is necessary to check the results of the rational method by experimental determinations. Some important structural problems, notably the design of columns, still are, and will perhaps remain, dependent on experimental data for their solution. Other problems, particularly those involved in masonry construction, depend largely, if not wholly, on empirical solutions. However, most of the important structural problems admit of solutions which are largely rational.

It is important that the student keep these methods and the distinction between them clearly in mind. Empirical methods are to be avoided whenever possible, but often rational and empirical methods must be combined. It is wise to keep clearly in mind which processes rest on a solid rational basis and which depend on the less trustworthy empirical knowledge.

4. Calculations. A. QUANTITIES. In the problems that follow the quantities dealt with are chiefly those of force and space, and in English speaking countries they are measured usually in pounds and inches, though often such units as tons and feet may be used. Time is dealt with to a limited extent.

With the exception of abstract numbers, each quantity has two characteristics: one qualitative, the other quantitative. Thus the expression *four feet* contains the two ideas, one (four) quantitative, the other (feet) qualitative.

When dealing with such quantities one should keep in mind the rule of arithmetic, that only quantities that are alike qualitatively (of the same denomination) may be added or subtracted. However, like or unlike quantities may be multiplied or divided. Such an operation consists in multiplying or dividing the quantitative and qualitative parts separately. The resulting quantity will be a derived one, unlike either of its factors, but partaking of the nature of each. Thus 4 ft. × 4 ft. = 16 sq. ft.*; 5 miles ÷ 4 hours = 1.25 miles per hour;† and 100 pounds ÷ 10 sq. ft. = 10 pounds per sq. ft.‡

B. Type of Problem. The problems that are to be solved are very similar to those of simple algebra and trigonometry. Certain quantities are known, others unknown; and there is a law (or laws) which governs their relations. It is then only a question of stating the known relations, in the form of equations, and solving these equations for the unknowns.

The student is warned that in some of the problems data is furnished which is entirely unnecessary to the solution. In other problems necessary data is omitted, and perhaps no definite solution is possible. This is in agreement with working conditions. One of the important points in structural designing is to sift the available data, separating the necessary from the unnecessary. Moreover the kind of knowledge that is valuable is the kind which cannot be shaken from its foundations by a few curiously stated problems.

C. Degree of Accuracy. The degree of accuracy required in structural computations is, relatively, not great. Experimental determinations of strength, modulus of elasticity, and similar quantities can rarely be made definite to more than two significant§ figures. The loads to be carried must be estimated

* Sometimes written 16□′ or 16 ft.^{2}

† May be written, 1.25 $\frac{\text{miles}}{\text{hour}}$.

‡ May be written, 10 lbs. /□′ or 10 lbs./ft.^{2} or #/□′.

§ In a number which begins or terminates with a group of ciphers, these ciphers are not significant figures. Thus each of the numbers 973,000,000 and 0.00973 is a number of three significant figures. Moreover, if each of these numbers is written as an approximation of a number of four or more significant figures, there is the same degree of accuracy in each.

rather than definitely determined. Therefore computations employing these quantities are usually indefinite beyond the third significant figure.

Slide-rule computations can easily be made accurate to within a fraction of one percent; and are, therefore, sufficiently accurate.

If, on the whole, the degree of accuracy in any computation is as great as that of the data on which it is based, the result will be satisfactory.

D. Analysis. Much confusion and loss of time can be avoided, and useful habits can be formed, by careful attention to the analysis of conditions and to methods of procedure. Before any figures are set down the conditions of the problem should be completely visualized. In nearly every case a diagram or sketch will be useful. Next the known and unknown quantities should be clearly recognized and the laws which connect them should be determined. Before proceeding with the solution, it is wise to form a mental estimate of the probable result, both quantitatively and qualitatively. When all this has been done, and not until then, may the actual computations be performed with profit, and with a reasonable assurance of success.

E. Presentation. Ordinarily too little attention is paid to the way in which computations are set down on paper. Slovenly methods of presentation usually indicate or lead to slovenly mental processes. The habit of performing parts of computations on any scrap of paper that may be at hand is a thoroughly bad one. Good habits will be formed if a notebook is kept in which all problems are set down in a clear and concise manner. Begin with a diagram which will record and clarify all given data. Then proceed to the solution which should be arranged according to a carefully considered and sequential scheme. The computations should show every operation, including even the extended multiplications, divisions, etc., unless a slide rule is used.

F. Checks. When a result has been obtained it should always be checked by comparison with some known condition taken preferably from every-day experience. Thus if a computation shows that the strength of a $3/8''$ rope is 50 lbs., while we know

from experience that it will easily support the weight of a man, there is reason to suspect an error in the figures.

By forecasting the results of each computation before any figures are set down, and by applying the check of reasonableness after results are obtained, the student will go far toward developing good judgment, which is, in many ways, more important than accurate knowledge or facility.

G. The Slide Rule. The student should begin at once to use a slide rule for performing routine computations. Until some facility is attained, the operation will prove more laborious than the ordinary extended computations; but after a little practice it will be found that computations can be performed much more quickly, and the mind, relieved of the burden of routine, will carry more easily the essence of the problem.

5. Historical Note. While literature is full of references to the remarkable scientific achievements of the ancients and of the mediaeval builders, the simplest facts of history show that their results must have been obtained by purely empirical processes. The simplest laws of statics were being formulated while Brunelleschi (1420–1464) was building his famous dome on the Duomo in Florence.

The first accurate knowledge of the elasticity of materials dates from the announcement of Hooke's law, in 1678. It was not until early in the nineteenth century that the modulus of elasticity was introduced into scientific determinations. From this time on development was rapid. By 1850 machinery and methods for testing materials had been developed, and some progress had been made in accumulating data.

The development of the steel industry, which is practically confined to the past fifty years, has encouraged and even necessitated a remarkable growth in our knowledge and our equipment for handling structural problems.

PROBLEMS

Note. Before starting work on the problems and frequently thereafter, the student should refer to Chapter I and particularly to § 4. A notebook, carefully prepared and kept corrected to date, is of the utmost importance.

The results obtained in a certain problem will be frequently used as data in a problem which follows.

1. In each of the following expressions, determine the resulting quantity in both the qualitative and the quantitative sense.

(*a*) $(10' \times 4'' \times 30\tfrac{1}{2}' \times 8'')^{1/2}$

(*b*) $\dfrac{90°}{8 \text{ sec.}}$

(*c*) $(10' \div 8 \text{ sec.}) \div 4 \text{ sec.}$

(*d*) $9 \text{ lbs.} \div 6''$

(*e*) $\dfrac{9'' \times 9'' \times 1\tfrac{1}{2}'}{2 \text{ hrs.}}$

(*f*) $(8' \times 8' \times 25')^{1/3}$

(*g*) $(8' - 4 \text{ sec.})^{1/2}$

(*h*) $9 \text{ tons} \div (6' \times 8')$

2. In each of the following expressions, determine the resulting quantity in both the qualitative and the quantitative sense.

(*a*) $9 \text{ meters} \times 6 \text{ ft.}$

(*b*) $(8 \text{ lbs.} \times 6') \div 12 \text{ sec.}$

(*c*) $(8 \text{ tons} \times 500 \text{ miles}) \div 24 \text{ hrs.}$

(*d*) $(12'' \times 6' \times 3'' \times 10')^{1/2}$

(*e*) $(8' - 4'')\ (24 \text{ lbs.})$

(*f*) $25 \text{ tons} \div (6'' \times 12')$

(*g*) $(8'' - 5 \text{ sec.})\ (6'' - 3'')$

(*h*) $(24 \times 6' \times 8' \times \tfrac{1}{2})^{1/2}$

CHAPTER II

FORCE, MOTION AND EQUILIBRIUM

6. Introduction. The essence of a machine is motion. The essence of a structure is rest. Each of them must fulfill its purpose by exerting or resisting force.

Mechanics, which deals with the action of forces on bodies, is the fundamental science of either mechanical or structural engineering. It has two main divisions:

(1) *Dynamics,* which treats of forces producing motion and is of primary importance to the mechanical engineer.

(2) *Statics,* which treats of bodies at rest under the action of forces, and which is of primary importance to the structural engineer.

We are so accustomed to the apparent quiescence and immutableness of structures that we are apt to forget that every structure maintains itself only by constant opposition to the forces which tend to bring it down. Gravity, wind, change in temperature, applied loads; some or all of these are constantly at work. It is the problem of the structural engineer so to design his structure that at all times it may resist these forces and hence maintain its static quality.

It follows that statics will be our chief concern. Occasionally, however, there is a case, usually that of moving loads, which involves dynamics. Moreover, since rest (a state of rest) has no significance except in contrast to motion, we must start our study of statics by considering motion and the laws which govern motion. (Also see Chap. XXIII.)

7. Laws of Motion. The following laws of motion are essentially the same as those first stated by Sir Isaac Newton. They are generalizations from experience and observation, and they depend for their validity principally upon the fact that human experience cannot be quoted to the contrary.

(1) *Every body continues in its state of rest or of motion in a straight line, except in so far as it is made to change that state by external forces.*

It is a matter of common experience that it takes force to move a body which is at rest. The converse proposition, namely, that every body at rest tends to remain at rest if it is not acted upon by some force, may be taken as axiomatic.

That all moving bodies tend to move in a straight line is witnessed by such familiar examples as the skidding of a motor car, the flying of mud from a revolving wheel, etc. In general this first law can find ample verification in common experience.

(2) *A body acted upon by a force receives an acceleration in the direction of the force which is proportional to the force and inversely proportional to the mass of the body.*

If a body is already moving at a uniform velocity, the application of a force will tend to increase this velocity if the force acts in the direction of motion, or to decrease the velocity if the force acts contrary to the direction of motion (i.e., the force gives to the body a positive or negative acceleration).

The action of the force of gravity as applied to falling bodies is a familiar illustration of this fact. Its action is continuous and the resulting velocity increases constantly, being 32.2 feet per second at the end of the first second, 64.4 feet per second at the end of the second second, etc. The *rate of increase* of velocity (the ***acceleration***) in this case is 32.2 feet per second per second. In the case of a body thrown upward the acceleration tends to reduce the velocity, being $-$ 32.2 feet per second per second.

The behavior of the balls in billiards, tennis, or baseball, or of a hockey puck, is a common illustration of the working of this law.

(3) *The change of motion produced by two or more forces acting on a body will be the same whether the forces act simultaneously or in turn, provided each force acts for the same length of time in both cases.*

This principle is frequently illustrated by the physical experiment in which two bodies, one dropped vertically and the other

projected horizontally from the same point, are found to reach the same horizontal surface in the same length of time. Also, if the block A in Fig. 1 is acted upon by a force Y, giving it a velocity which will move it from A to C in a given time, and then by a force Z which will move it from C to B in the same length of time, the simultaneous application of Y and Z will give it a velocity which will move it directly from A to B in the same length of time as each of the former motions.

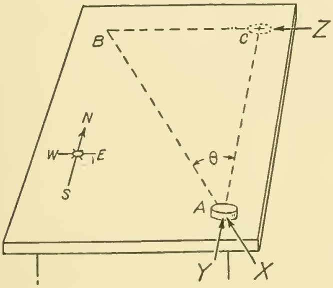


Fig. 1

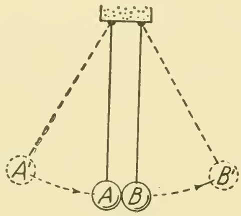


Fig. 2

(4) *Any action (force) of one body on another is accompanied by an equal and opposite reaction of the second on the first.*

The recoil of a gun is the reaction to the force acting on the projectile. A further illustration is given by the familiar physical experiment illustrated in Fig. 2. Two elastic balls, alike in all respects, are suspended from cords so that they just touch one another. Now let the ball A be swung into the position A' and then allowed to swing back till it strikes B. If there is no loss due to friction or imperfect elasticity, the ball A will be stopped at A, and B will swing to B' which is at the same height as A'. In this case the action of A is transferred to B and causes B to move. The reaction of B on A causes A to stop.*

8. Reactions. In the cases quoted in § 7 (4), action and reaction are essentially the same except that one is in the nature of a cause; the other being an effect.

In the class of problems with which we are chiefly concerned

* The definition of force, as given in *The Mechanics of Engineering* by I. P. Church, is as follows: "A force is one of a pair of equal, opposite and simultaneous actions between two bodies by which the state of their motions is altered or a change of form in the bodies themselves is effected."

there is an important distinction between a force and the accompanying reaction. Therefore the word reaction will come to be used in a somewhat special sense which can be illustrated by the following cases.

Any object lying on a table is acted upon by the force of gravity. It would move unless some force were resisting the force of gravity. The table reacts to the load and prevents motion. The end of a beam or truss resting on a wall produces a pressure on the wall due to the force of gravity. The wall reacts and supports the load. In each case the force is active while the reaction is a passive resistance set up by the force. In the case of the beam resting on the wall, the removal of the wall from under the beam allows the force of gravity to come into play and the beam falls. The removal of the beam from the wall produces no *visible* effect on the wall.

For the purposes of this book we may then recognize a reaction as the passive resistance set up in one body by the action of another body on it.

9. Force Characteristics. Every force has three characteristics: (*a*) its amount, (*b*) its direction, and (*c*) its point of application. These characteristics and the effect of each in producing motion are well illustrated when a cue strikes a billiard ball. The ball will move away from the cue: (*a*) at a definite speed, (*b*) in a definite direction, and (*c*) with or without a definite spinning motion. These characteristics of motion will be controlled by (*a*) the force of the stroke, (*b*) the direction of the stroke, and (*c*) the point where the stroke is applied to the ball.

For the present we will deal only with problems involving the amount and the direction of forces and the consequent motion of ***translation*** (i.e., motion which transfers a body from one point to another along a straight line). The point of application of forces and the consequent spinning or rotative motion (***rotation***) produced are considered in Chapter V.

The ***amount*** of a force is measured by comparison with the force of gravity acting on a known body, and is expressed in pounds, tons, grams, etc. Thus a force of 10 lbs. is ten times as great as the force of gravity acting on a 1-lb. weight.

The *direction* of a force is measured by reference to arbitrarily chosen axes (usually rectangular). Thus a force is said to act vertically, or horizontally, or at 45° to the vertical, etc. In Fig. 3 the force C' is a force of 50 lbs. acting at $\theta°$ to the horizontal. It should be noted, however, that the angle θ alone does not indicate whether the force is directed upward and to the right or downward and to the left. There is also needed an arrow to show in what *sense* the force acts along the line. From this point of view the arrow really indicates whether the force is a force of + 50 lbs., or − 50 lbs., acting in a *direction* which is given by the line.

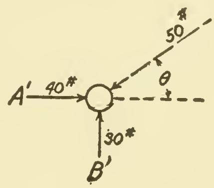


FIG. 3

10. Classification of Forces. For the sake of simplifying the study of the relations and effects of forces, it will be found convenient to establish definite groupings based on the relations between the lines along which the forces act.

When all the forces in a given problem act along lines that lie in the same plane (as in Fig. 1), the forces are said to be ***coplanar forces.*** Again, when all the lines of action meet at a common point (as in Fig. 13), the forces are said to be ***concurrent.*** Under the opposite conditions forces are classed as ***non-coplanar*** and ***non-concurrent.*** These classifications are independent. Thus we may have a set of forces which are coplanar and non-concurrent, or non-coplanar and concurrent, etc.

For the present we shall deal only with concurrent coplanar forces, leaving the other cases for later study.

11. Equilibrium. Any body which is at rest under the action of forces is said to be in ***equilibrium.*** In our study of statics, equilibrium is the normal condition, since it is the proper condition for the parts of a structure.

12. Methods. Two general methods are available for handling problems dealing with forces.

The Analytic Method uses the abstract processes of algebra and trigonometry, applied to the laws of motion and equilibrium.

The Graphic Method makes use of lines to represent forces and partakes more of the character of geometry. The fundamental principle of the graphic method lies in the fact that a straight line has three characteristics: length, direction, and position; and these may be used to represent the three force characteristics: amount, direction, and point of application. The graphic method is, for most people, so much the easier to understand and to apply, particularly in the simple problems which are first taken up, that there is a tendency to neglect the other method of solution. However, cases will constantly arise where the analytic method is much less laborious as well as simpler.*

It is necessary for the student to comprehend thoroughly both of the above methods so that he may make an intelligent selection, suited to the cases that arise. In fact the usual solutions for many problems are partly analytical and partly graphical.

PROBLEMS

NOTE. In the following problems, let air resistance be neglected.

1. How far will a freely falling body drop in 12 seconds?
2. A stone is dropped from a bridge 200 ft. high. How long will it be in falling?
3. A bullet is shot vertically upward with a velocity of 500 ft. per sec. How far will it rise?
4. A bullet is shot vertically upward with a velocity of 500 ft. per sec. from the top of a building 500 ft. high. It rises then falls to the ground. If sound travels 1100 ft. per second, how long after the shot is fired will the bullet be heard striking the ground by an observer on top of the building?
5. A bullet is shot horizontally from the above building with a velocity of 500 ft. per sec. How far from the base of the building will it strike the ground?
6. In problem 5 what kind of a curve will the bullet follow?
7. A boat is rowed directly across a stream which is 400 yds. wide. The rate of rowing is 4 miles per hour. The current has a velocity of 2 mi. per hr. to midstream and then 6 mi. per hr. to the opposite bank. (*a*) What will be the time of crossing? (*b*) How far downstream will the boat land?
8. A boat is rowed directly across a stream which is 400 yds. wide. It lands 150 yds. downstream from the starting point, at the end of 5 min. (*a*) What (in miles per hour) is the velocity of the current? (*b*) What is the rate of rowing? (*c*) What is the velocity of the boat?
9. In problem 8, if the stream velocity is zero at the start, increasing uniformly to 6 mi. per hr. at opposite bank, what will be the path of the boat and how far downstream will it land?

* For a more complete discussion of the graphic method, see §§ 247—249.

CHAPTER III

CONCURRENT COPLANAR FORCES

13. Introduction. Since the lines of action of concurrent forces meet at a common point, all such forces have the same point of application. Thus one of the possible causes of complexity of relationship is eliminated. Consequently the problems arising from the relations of concurrent forces are quite simple and the solutions are direct. Just as plane geometry is more simple than solid, because of there being two dimensions with which to deal instead of three, so in problems dealing with forces, the simpler solutions are found where the number of possible variables is least.

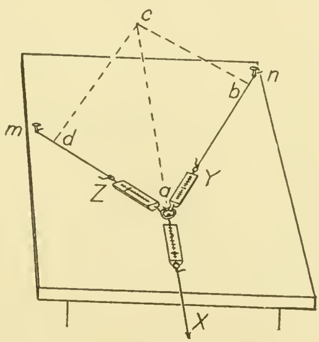


FIG. 4

14. The Fundamental Relations. A common experiment which shows the fundamental relations between concurrent forces is shown in Fig. 4. Three spring balances are rigged to a ring and two of them are fastened also to nails in the table top. Now any pull at X will set up other pulls at Y and Z. These pulls are concurrent forces since their lines of action meet at the ring and the relations that are found to exist between them are characteristic of concurrent forces in general.

As a matter of fact, if any pull is exerted at X, Fig. 4, and the resulting pulls on Y and Z are noted, the three readings will

always be found to bear a definite relationship to one another. This relationship is expressed in Fig. 4 by the parallelogram *abcd* which is laid out with its sides parallel to Z and Y and its diagonal of such a length as will represent (at some definite scale) the pull on X. The lengths of the sides of the parallelogram (ad, ab) are then found to be equal (at the same scale as before) to the pulls on Z and Y.

Here we have a case of three concurrent forces in equilibrium. Obviously if any one of the forces is removed, the other two will cause the ring to move. Hence any one of the forces can be regarded as holding the other two in equilibrium. Such a force is said to be the ***equilibrant*** of the other two. A force which is equal and opposite to the equilibrant would produce the same *result* as the other two. Such a force is said to be the ***resultant*** of the two forces. Thus an equilibrant (sometimes called an ***anti-resultant***) and a resultant are equal and opposite forces whose characteristics can be determined by a principle known as the parallelogram of forces and which is illustrated by the above experiment. This principle may be stated:

> When two forces act at a point, the resultant force can be represented as to amount and direction by the diagonal of a parallelogram, the sides of which are parallel to and proportional to the two original forces.

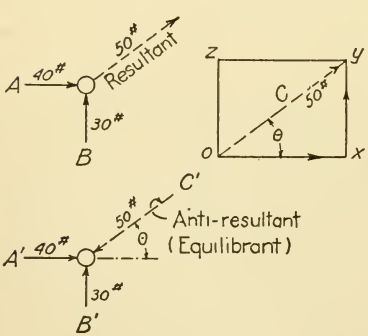


FIG. 5

A special case occurs when the two given forces act in the same straight line, either in the same or in opposite directions. The solution is obviously a simple matter of addition or subtraction.

Figure 5 shows the general relationship between two forces, their resultant and the anti-resultant or equilibrant. The forces A and B have as their resultant the force C whose amount and direction are determined from the parallelogram *oxyz*. Now

the two forces A and B can be considered as those forces which will have the same effect as C. When so considered they are called the ***components*** of C; the force A being the component in a horizontal direction while B is the component in a vertical direction. Thus if the component forces A and B act together on the body, their effect will be the same as if C acted alone. Components in directions other than those used in this case could as readily be determined by a similar procedure.

15. Applications. Let a force of 10 lbs., acting at an angle of 30° from the vertical, be given and let it be required to determine its horizontal and vertical components.* Let the given force be represented by AB and let the given angle be represented by $\theta = 30°$, Fig. 6. Then its components are represented by AC and CB. Solving the triangle, we find

$$AB = 10 = BC \div (\sin \theta) = 2BC,$$
$$BC = 5 \text{ lbs.}$$
$$AB = 10 = AC \div (\cos \theta) = AC \div 0.866,$$
$$AC = 8.66 \text{ lbs.}$$

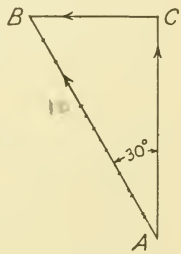


Fig. 6

The same result could have been obtained by laying out the triangle at scale and determining AC and BC by measurement.

Since the given force acts from A toward B, its components must act from A toward C and from C toward B, as shown by the arrows.

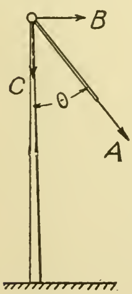


Fig. 7

Figure 7 shows a pole set in the ground. A rope fastened to its top produces a pull A which is inclined at the angle θ to the pole. This force has a component B in a horizontal direction which tends to bend the pole and a component C in a vertical direction which produces compression in the pole. If values are assigned to A and θ, the values of B and C can be determined as in the preceding case.

* Hereafter H and V will be used as abbreviations for horizontal and vertical when used in qualifying the components of a force.

Next let it be required to find the resultant of two forces which act at angles other than right angles, as in Fig. 8*A*. The solution varies from the above only in that the trigonometric solutions for irregular triangles are used instead of those for right triangles. Thus let the forces C and D and the angle between them be given, as shown in the figure, and let the amount and direction of the resultant be required.

Draw the parallelogram *oefg* with its sides parallel to the given directions of C and D, as in Fig. 8*B*. We know that $og = 60$ lbs., $gf = 30$ lbs., and (since $eog = 60°$) $ogf = 120°$. Then we have

$$(1) \qquad fh = of \sin \theta = 30 \sin 120° = 25.99,$$

$$(2) \qquad oh = of \cos \theta = 60 + 30 \cos 120° = 75.$$

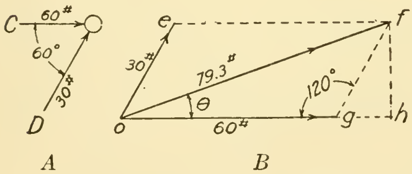


Fig. 8

Then by division we get

$$\tan \theta = \frac{fh}{oh} = 0.346,$$

$$\theta = 19° \; 7',$$

and from equation (1),

$$of = 79.3 \text{ lbs.}$$

Therefore the resultant of C and D is a force of 79.3 lbs. acting at 19° 7′ from the horizontal.

A graphic solution for this case, similar to the one in Fig. 5, can be performed easily without further explanation.

16. Body under Action of Three or More Forces. In this case the forces concerned may have a resultant or they may be balanced, producing equilibrium. In case there is a resultant its amount and direction may be determined from the following considerations.

A. Analytic Solution. Either of the two following solutions may be used. Neither one is more than a slight extension of the principles of §§ 13 to 15.

(1) *Successive combination.* Three forces being given, as in

Fig. 9, the resultant of any two can be found by the method of § 15. The resultant thus found can then be combined with the third force to get the resultant of the three given forces. Thus in Fig. 9*B*, *D* is the resultant of *A* and *B*. By combining *D* and *C* as in Fig. 9*C*, we get *R*, which is the resultant of *A*, *B*, and *C*.

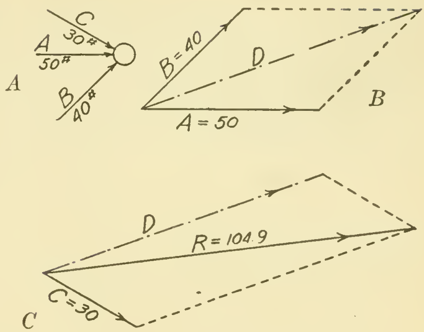


FIG. 9

(2) *Summation of components.* Let each of the forces in Fig. 9 be replaced by its H and V components, as shown in Fig. 10. Let all V components acting upward be called positive and those acting downward negative. Let all H components acting toward the right be called positive and those acting toward the left negative. Now by summation there is found a resultant H component of + 104 lbs. and a resultant V component of + 13 lbs. The resultant of the three given forces is then the force *R* shown in the figure. Its amount is $\sqrt{\overline{13}^2 + \overline{104}^2} = 104.9$ lbs. and its inclination toward the horizontal is the angle whose tangent is 13/104, or 7° 10′.

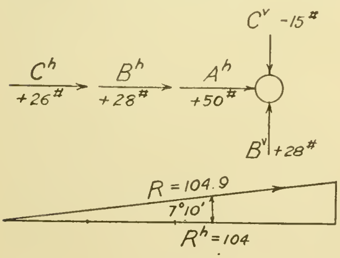


FIG. 10

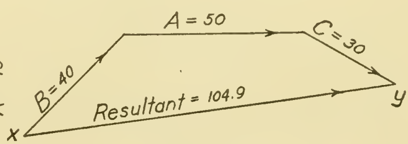


FIG. 11

B. GRAPHIC SOLUTION. If the forces in Fig. 9 are laid off at scale and end to end, as in Fig. 11, the direction and amount of the resultant is given by a line drawn from the point of beginning x to the end of the last force y. In laying out the forces, care must be taken to see that the arrows on the forces read continuously from x to y, never reversing directions. This merely means that a proper graphic summation is performed.

The nature of each of the above solutions is such that it can be extended readily to cover any number of given forces. It is evident also that if three or more forces are in equilibrium, any one of the forces is the anti-resultant of all the others.

PROBLEMS

NOTE. In the following problems, prepare an analytic and a graphic solution for each.

1. A surveyor starts from a bench mark and runs north 37° east, 1750 ft.; from there he runs west 10° south, 500 ft.; and from there south 25° west, 1000 ft. What is the distance and direction from the last point to the starting point? (Work this by computing component distances.)
2. Refer to Fig. 7. If $A = 100$ lbs., what are the values of C and B when θ has each of the following values: 10°; 30°; 45°; 90°; 135°; 180°; 315°?
3. A horse draws a wagon weighing 3000 lbs. up a 20° slope. What force must he exert?
4. In Fig. 4, if the angle $a = 80°$ and $acb = 20°$, what will be the readings of Z and Y when that on X is 70 lbs.?

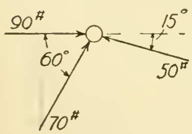


FIG. 12

5. In Fig. 4, the readings are $X = 80$ lbs., $Y = 100$ lbs., $Z = 50$ lbs., what are the angles?
6. In Fig. 4, if the angle $b = 75°$, and the reading on Z is 50 lbs. and on Y 100 lbs., find the reading on X and the angles a and c.
7. Find the resultant of the forces in Fig. 12 by the successive combination method.
8. Repeat problem 7 using the summation of components method.
9. Draw a curve to show the variation in the horizontal and vertical components of a force of 100 lbs., as the direction of that force varies from horizontal to vertical.

17. Conditions of Equilibrium. Since the parts of almost any structure are at rest and should remain so under all conditions, it is very important that the student should understand thoroughly the conditions under which equilibrium may exist. Moreover, he should have them so formulated that they will remain in his mind and be available for constant use.

A. ANALYTIC METHOD. From § 16 it can be seen that equilibrium may not exist except when the sum of the H components of all the forces acting on the body is zero and the sum of the V components of all the forces is also zero.

It will be noted that this involves two *separate* summations and *each* must be equal to zero. Thus in Fig. 13 $A + B$

= + 45 lbs. − 45 lbs. = 0, but evidently there is a resultant of these two forces. If now C is added to the system, $A + C = +45 - 45 = 0$. There is now no resultant V component; but B is unbalanced, and it is only when D is added, so that the summations become *separately* equal to zero, that equilibrium obtains.

The conditions which are necessary and sufficient to produce and maintain equilibrium with only concurrent coplanar forces acting, are:

1. *The sum of the H components of all the forces involved must be zero.*
2. *The sum of the V components of all the forces involved must be zero.*

These statements may be abbreviated into the form

(1) $$\Sigma H = 0,$$

(2) $$\Sigma V = 0.*$$

B. Graphic Method. If the forces in Fig. 13 are laid off at scale joining continuously, and each in the direction indicated by the arrow on it (as shown in Fig. 14), a closed polygon is the result. Moreover the same thing is bound to occur whenever a system of concurrent coplanar forces is in equilibrium. For in that case, since the summation of components must be zero, as shown above, a broken line representing all the forces is bound to end up at the point of beginning. This is independent of the number of forces or of their amounts and directions.

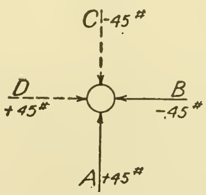


Fig. 13

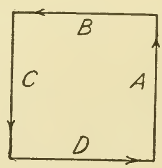


Fig. 14

A broken line laid off as described above, and which represents all the forces in a problem, is called the *load line* or the *force line*.

The sufficient and necessary condition for equilibrium of con-

* The character Σ stands for a summation, i.e., for an algebraic addition of all like quantities which enter into the problem.

current coplanar forces in a graphic solution is, then, that the load line must be a closed polygon. This statement is the graphic equivalent of equations 1 and 2 in (A).

18. Application. In Fig. 15*A* is shown a ring carrying a load of W lbs. and hanging on a cord. A force H is applied horizontally to the ring and causes it to swing out from the vertical. If H and W are constant, the ring will come to rest in some position, indicated by the (unknown) angle θ. Let it be required to determine θ in terms of H and W.

When at rest the ring is in equilibrium under the action of three forces, H, W and the pull in the cord P; as in Fig. 15*B*. The components of P are $P \sin \theta$ and $P \cos \theta$ as shown. Now if equilibrium exists, $\Sigma H = 0$ and $\Sigma V = 0$ (§ 17); hence we have

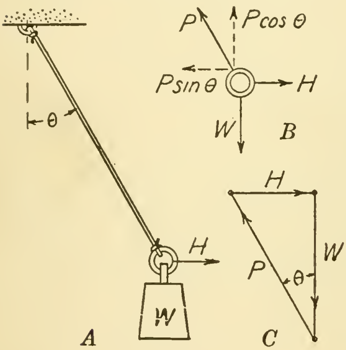


FIG. 15

$$H - P \sin \theta = 0,$$

$$P \cos \theta - W = 0,$$

or

$$(1) \qquad H = P \sin \theta,$$

$$(2) \qquad W = P \cos \theta.$$

Now by dividing (1) by (2), we get

$$\frac{P \sin \theta}{P \cos \theta} = \frac{H}{W}, \quad \text{or} \quad \tan \theta = \frac{H}{W}.$$

From this relation θ may be determined when H and W are known. After θ has been thus determined, P can be found from equation (1) or (2), or from the relation $P^2 = H^2 + W^2$.

An alternative solution, by the graphic method, is shown in Fig. 15*C*. The two known forces H and W are laid off at scale. Since the force P must be in equilibrium with H and W, it must be represented by the line P which closes the load line (§ 17). Therefore in Fig. 15*A* the string will come to rest in a position parallel to the line P in Fig. 15*C* and the angle θ, Fig. 15*C*, is the required angle.

Let the student now assume the force W as 1 lb., and determine how large H must be for several values of θ between 0° and 90°.

19. Summary—Concurrent Coplanar Forces. The problem of § 18 might be stated so that W and θ are known and H is required, or in other ways; but the principle of the solution would remain the same.

In any problem in concurrent coplanar forces, each force has two characteristics: amount and direction.* In § 17, two equations are established, based on the two conditions of equilibrium. These equations can be made to determine two unknown quantities and only two. Thus one force may be determined in amount and direction, or two in direction only, or two in amount only, or one in amount and one in direction.

PROBLEMS

1. Find the equilibrant of the forces in Fig. 16.
2. Find the resultant of the forces in Fig. 17. Use the graphic method. Then make new solution changing the order in which the forces are set down.
3. See Fig. 15A. Find θ when W = 81 lbs. and H = 46 lbs.
4. See Fig. 15A. Find H when W = 100 lbs. and θ = 46°.
5. See Fig. 15A. Find W when H = 1 lb. and θ = 5°.

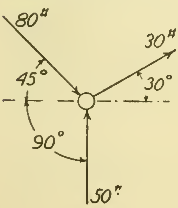


FIG. 16

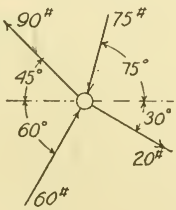


FIG. 17

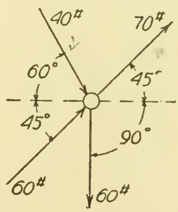


FIG. 18

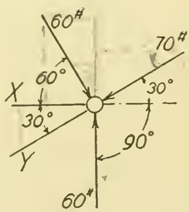


FIG. 19

6. See Fig. 15A. Let W = 1 lb. What pull, H, will cause θ to become 89°?

NOTE. Problems 7 to 10 shall be solved graphically. Scale, 1″ = 20 lbs.

7. Find the direction of a force of 80 lbs. and that of another force of 62 lbs. which may be used to equilibrate the forces in Fig. 18.
8. What forces acting on the lines X and Y, Fig. 19, will equilibrate the other forces?
9. Find the directions of two forces, one of 40 lbs. and the other of 60 lbs., that will equilibrate those shown in Fig. 20.
10. Find the direction and amount of an inclined force R and the amount of a horizontal force S which will equilibrate the forces shown in Fig. 16.

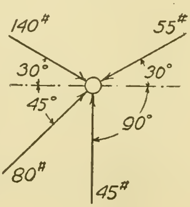


FIG. 20

* Since the point of application is common to all forces in such a problem it has no influence on the problem.

CHAPTER IV

FORCES AND STRESSES

20. Introduction.—Definitions. When a heavy weight is placed on a rope, as in Fig. 21, the rope becomes longer and thinner, while the strands can be seen to adjust themselves to their work. This change of shape is the visible evidence of a changing internal condition in the rope. This condition is due to the application of the load and is said to be a condition of stress. If the load is increased, the internal stress becomes more severe. This increase of stress will continue with an increasing load, up to the limit set by the strength of the rope. When the load becomes too great, the rope can no longer resist, because the internal stress becomes greater than the material can bear. In that case the rope breaks and motion ensues because the internal stresses no longer balance the external forces. That is to say ΣV is no longer zero (§ 17).

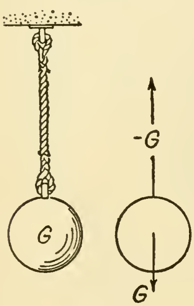


FIG. 21

From the preceding discussion the distinction between force and stress becomes clear. A force is external. A ***stress*** is the internal resistance set up by an (external) force.

In the case of most materials commonly used in construction, such changes in size and shape of parts under the action of external forces (loads) are not visible; but it has been amply proven by careful experiments that they do take place. The study of the effects of loading in producing changes in size and shape of structural parts will form the basis of future work. For the present it is sufficient to note the existence of this internal resistance to external force, which is commonly called stress, and to determine its amount without dealing with its effects.

21. Character and Amount of Stress. Stresses are measured in the same terms as forces, i.e., in pounds, tons, grams, etc.

Stresses are analogous to reactions (§ 8), in that they are not active forces but rather passive resistances called into play by the action of some external force.

When the line of action of an external force coincides with the axis of the supporting member (Figs. 21 to 24), the internal stresses are called ***simple axial stresses.*** In such a case, when the force tends to pull the supporting member apart, the member is said to be in ***tension,*** the stresses being called tensile stresses (Figs. 21 to 23). When the load tends to crush the member together, the stresses are those of ***compression*** (Fig. 24).

Besides these simple axial stresses of tension and compression we will have occasion to notice the case where equal and opposite forces act in adjoining planes. This case is called ***shear*** and is illustrated in Fig. 151 (compare to the action of a pair of shears). When the planes in which oppositely directed forces act are not adjacent as above (Fig. 85) or when the lines of action do not coincide with the axis of the member (Fig. 332), bending stresses are set up.

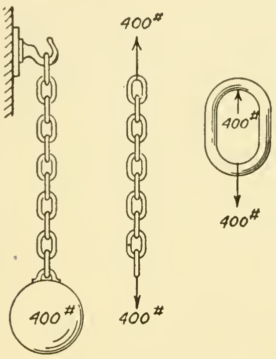
For the present our attention will be confined to the case of simple axial stresses of tension and compression, the cases of bending, shear, and combined stresses being taken up later.

22. Simple Axial Stress. It will be recognized that if any structure is in equilibrium as a whole, each and every part must be in equilibrium.* In Fig. 22*A* is shown a weight suspended on a chain and at *B* the same chain is shown, the action of the weight and the reaction of the hook having been represented by arrows. Here equilibrium exists; the chain as a whole being under the action of two equal and opposite forces. The lowest link is in equilibrium under the action of the weight and the reaction of the next higher link. Similarly every link is in equilibrium under the same equal and opposite forces, as shown at larger scale in Fig. 22*C*.

In Fig. 23 is shown a steel eyebar loaded and in equilibrium as a whole and in all its parts. If part *1* is conceived to be

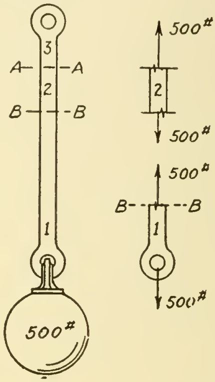
* This is true only when the presence of movable parts is taken as indicating the difference between a machine and a structure.

separated from part *2* (as in Fig. 23*B*), and if arrows are used to indicate the forces which must be acting on this part, there is found a downward force of 500 lbs. at the bottom which in the unseparated bar must evidently be balanced (since equi-



A *B* *C*

Fig. 22



A *B*

Fig. 23

librium prevails) by an equal upward force at *B–B*. This latter is the stress in the bar. Figure 23*B* shows part *2* and the stress which acts in the bar and which holds it in equilibrium.

It should be particularly noted that the arrows are here used to indicate the action of the *adjacent* part on the part under consideration. Thus in part *2* the arrow at *A–A* indicates the (upward) reaction of the pin (transferred through *3*); while the arrow at *B–B* indicates the (downward) pull of the load acting through *1*. Part *1* pulls downward on *2* while *2* pulls upward on *1*. Pulling or tensile stresses are then indicated wherever the arrow thus determined acts, *away from* a section cut through a body under stress.

Figure 24 shows a steel column similarly treated. It will be noted that the compressive stresses here dealt with must be indicated by arrows pointing *toward* the section.

In *every* case the direction of the stress arrow is determined from that of the force arrow.

What precedes may be summed up as follows:

(*a*) A *stress* is an *internal* resistance to an external force.

(*b*) In the case of simple axial stresses, the amount of the stress on any section through a body is equal to that of the applied force.

(*c*) The character of the stress (whether tension or compression) is determined by the direction of the applied force.

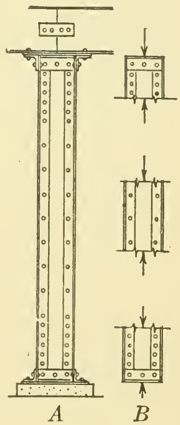


FIG. 24

23. Free-Body Method. The principles of §§ 21–22 may be used to determine the internal stresses in any structure whose members are subject to simple axial stress only,* the loads and reactions being known. The general method used is to conceive one part of the structure to be cut away from the adjoining parts and moved to a free position in space.

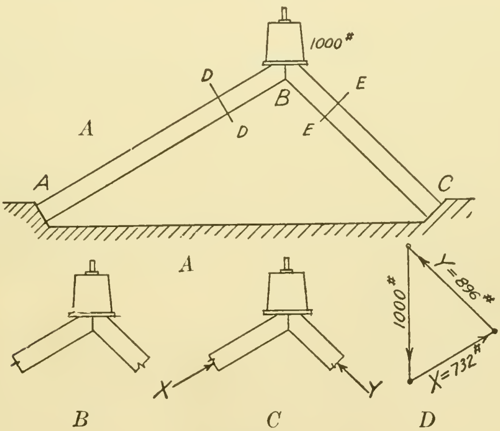


FIG. 25

* A body which is subjected to simple axial stress only is sometimes called a *two-force piece.* A chain or cable is, by its very construction, necessarily a two-force piece. An eyebar (Figs. 23 and 51) which receives its loads and reactions from pins through the eyes is also a two-force piece. Thus a truss or other frame composed wholly of pinned connections, and which is loaded only at the joints, will be stressed in all of its members, either in tension or compression, as each member is a two-force piece (Fig. 40). Small trusses and frames which are not so connected, but are fastened together with more or less stiffness in the joints, (Fig. 36) are commonly treated *as if* they were composed of two-force pieces. The computations are thus much simplified and the errors introduced are small. In general, and unless otherwise stated, problems dealing with framed structures will be treated, hereafter, as if all members were two-force pieces.

Thus in Fig. 25*A* the timbers *AB* and *BC* support the 1000 lbs. load and it is required to find the stresses set up in the timbers by the load. Imagine the timbers to have been cut along the planes *D–D* and *E–E* and the part thus freed to be moved to *B*. The ***free body*** is not in equilibrium since it is acted upon by a single force, viz., the 1000 lbs. load. However, in its *original* position *as a part of the structure* in Fig. 25*A*, it *was* in equilibrium. Therefore there must have been acting on the cut sections (*D–D* and *E–E*) such stresses as could hold the 1000 lbs. load in equilibrium. When the upper part is cut loose and made a free body, these stresses become *external forces, acting on the free body* and holding it in equilibrium as shown by the arrows in Fig. 25*C*. This last drawing shows a body which *may* be in equilibrium and which *will be* if the forces *X* and *Y* are properly determined. These forces are simple axial stresses in the timbers and their directions are therefore known. Their amounts may then be determined as described in § 18 and as shown in Fig. 25*D*. The arrows on these forces, as determined by Fig. 25*D*, when applied to the free body, Fig. 25*B*, act *toward* the cut sections and hence indicate compressive stresses.

We now know that the free body, Fig. 25*B*, will be in equilibrium if compressive stresses of 732 lbs. and 896 lbs. are present in the timbers of Fig. 25*A*, and we therefore conclude that when the timbers are carrying the load of 1000 lbs., they will be subjected to stresses of these amounts.

A further application of this method is given in Fig. 26*A*. Let it be required to find the stresses in the cords. In Fig. 26*B*, the ring is shown as a free body under the action of a downward force of 100 lbs. and the supporting pulls of the strings. These three forces are in equilibrium and are known in direction while one of them is also known in amount. In Fig. 26*C*, the line *ab* is drawn to represent the 100 lbs. load. This line, together with lines parallel to the other two forces, must form a closed polygon (§ 17). Therefore the lines *ac* and *bc* are drawn parallel to the forces concerned and the triangle thus formed is solved either by the analytic or by the graphic method (§ 18). The arrows on

the lines *bc* and *ca* are determined by that of the known force *ab*; they must be such as to read continuously and in the same direction: *a* to *b*, *b* to *c*, and *c* to *a*. When these are transferred to the free body, they act away from the cut sections and hence indicate tension.

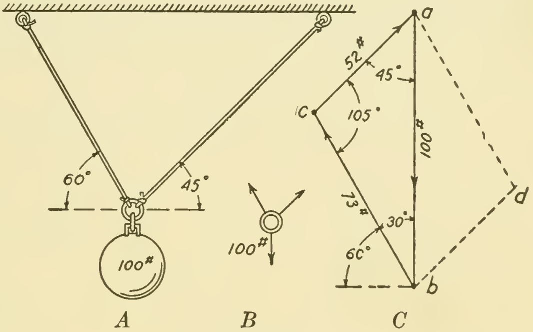


Fig. 26

Had the lines representing the stresses in the ropes been laid off from *ab* in a different *order*, the force triangle *abd* would have been constructed and its solution would have given the same results.

24. Summary. The free-body method for determining stresses is perhaps the most useful single idea in the study of statics. No satisfactory progress can be made until it is thoroughly mastered. Therefore the following summary may prove useful.

(1) If any part of a structure is cut away from the rest and made free, the stresses within the original structure become forces acting on the cut sections of the free body.

(2) These forces form a system in equilibrium.

(3) If there is at least one known force and not more than two unknowns, the amounts of the unknowns may be determined as in § 18.

(4) The arrows on the lines representing the unknown forces will be determined from that on the known force.

(5) The forces on the free body having been determined on the assumption that equilibrium exists, it is evident that in the

structure from which the free body was cut these forces exist as stresses on the cut sections.

25. A Derrick. Let it be required to determine the character and amount of stresses in each member of the derrick shown in Fig. 27*A*. For the purposes of this computation the derrick has been made of eyebars and therefore all members are two-force pieces, as explained in the note on page 27.

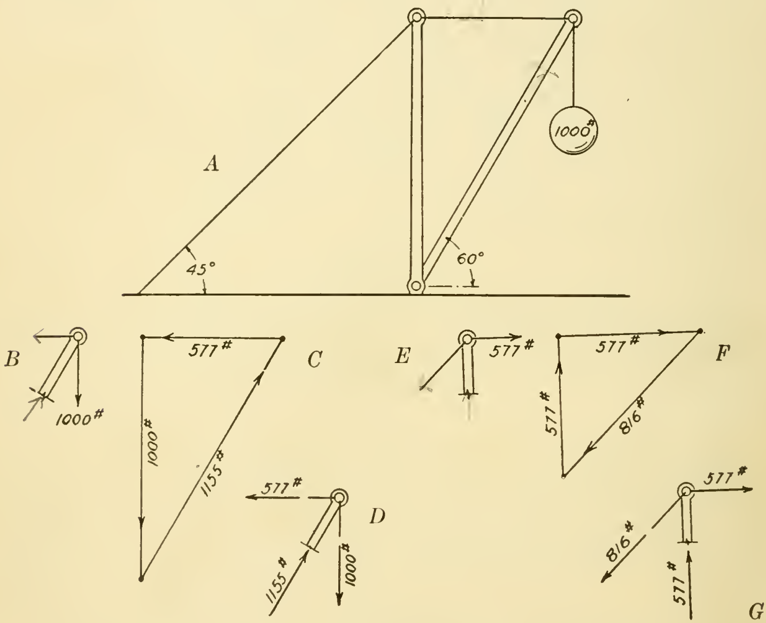


FIG. 27

If the boom, top-stay, and load are cut away from the rest, and treated as a free body, they will appear as in Fig. 27*B*. The force diagram for this free body is shown in Fig. 27*C*. This triangle may be solved analytically (using the known dimensions of the derrick to establish the angles of the force triangle) or graphically. The directions of the arrows may be determined as in § 23. When these forces are applied to the free body, as in Fig. 27*D*, we find that the boom is in compression (1155 lbs.), and the top-stay in tension (577 lbs.).

It is now possible to cut loose the top-stay, back-stay, and mast as a free body, as shown in Fig. 27*E*. Note that the stress in the top-stay is the 577 lbs. of tension determined in the previous solution. Since it is tension it must act *away from the cut section*, as shown.* With this known force as the initial line, the force diagram, Fig. 27*F*, is drawn and the stresses in mast and back-stay are determined as shown in Fig. 27*G*.

PROBLEMS

1. Find the stresses in the eyebar frame shown in Fig. 28.
2. Find the stresses on the blocks *W*, *X*, *Y*, *Z*, Fig. 29.

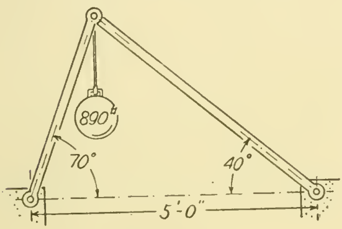


FIG. 28

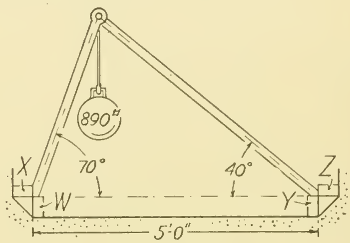


FIG. 29

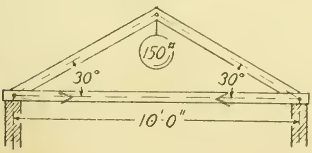


FIG. 30

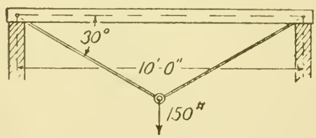


FIG. 31

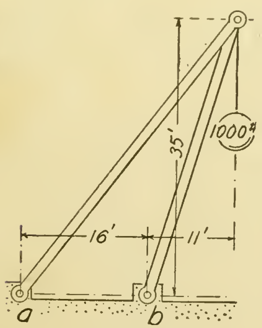


FIG. 32

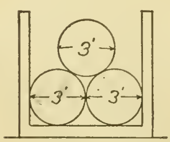


FIG. 33

* Students are frequently confused to find this arrow pointing in the opposite direction from the one in Fig. 27*D*, which represents the same stress. A review of §§ 21–22 should clear up this point.

3. Find the stresses in the members of the frame in Fig. 30.
4. Find the stresses in the members of the frame in Fig. 31.
5. Find the stresses in the members of the frame in Fig. 32.
6. Three equal cylinders are contained within a box as shown in Fig. 33. Each cylinder is 3′0″ in diameter and 4′0″ long and weighs 100 lbs. What is the direction and amount of the pressure on the sides and bottom of the box? Neglect friction.
7. In Problem 6, let the box be 7′6″ wide, inside. Graphic solution.
8. A wheel 3′0″ in diameter stands in a vertical plane and carries a vertical load of 500 lbs. on its axle. What horizontal force, applied to the axle, is needed to start the wheel over a block 6″ high?

26. Space and Force Diagrams. In the preceding example, Figs. 27 *A*, *B*, *D*, *E*, and *G* are space diagrams. They represent the derrick or some part of it at a scale of feet and inches. Where forces are shown on these diagrams no attempt is made to indicate them at scale but only to indicate their presence and direction, the amounts being in figures. On the other hand, Figs. 27*C* and 27 *F* are force diagrams drawn at a scale of pounds and indicating the forces at scale in their relative amounts and directions but without regard to position.

Students frequently attempt to combine force and space diagrams into one drawing. This inevitably leads to confusion and error. A little thought will make it evident that two kinds of diagrams are necessary to the solution of such a problem, and that this necessity lies in the nature of the quantities involved in the problem.

27. A Truss.* Let it be required to find the stresses in the various members of the truss in Fig. 34*A*, produced by the given loads.

If the peak joint is cut loose and shown free, as in Fig. 34*B*, it will be seen that we have one known and three unknown forces in the problem. Hence no solution is possible, as pointed out in §§ 19 and 24. The same state of affairs would be found if either of the joints marked *X* or *Y* were tried. Therefore the solution must start with one of the end joints.

The end joint is shown as a free body in Fig. 34*C*. The known

* In this solution the truss is assumed to be so connected that its members are all two-force pieces as explained in the note on page 27.

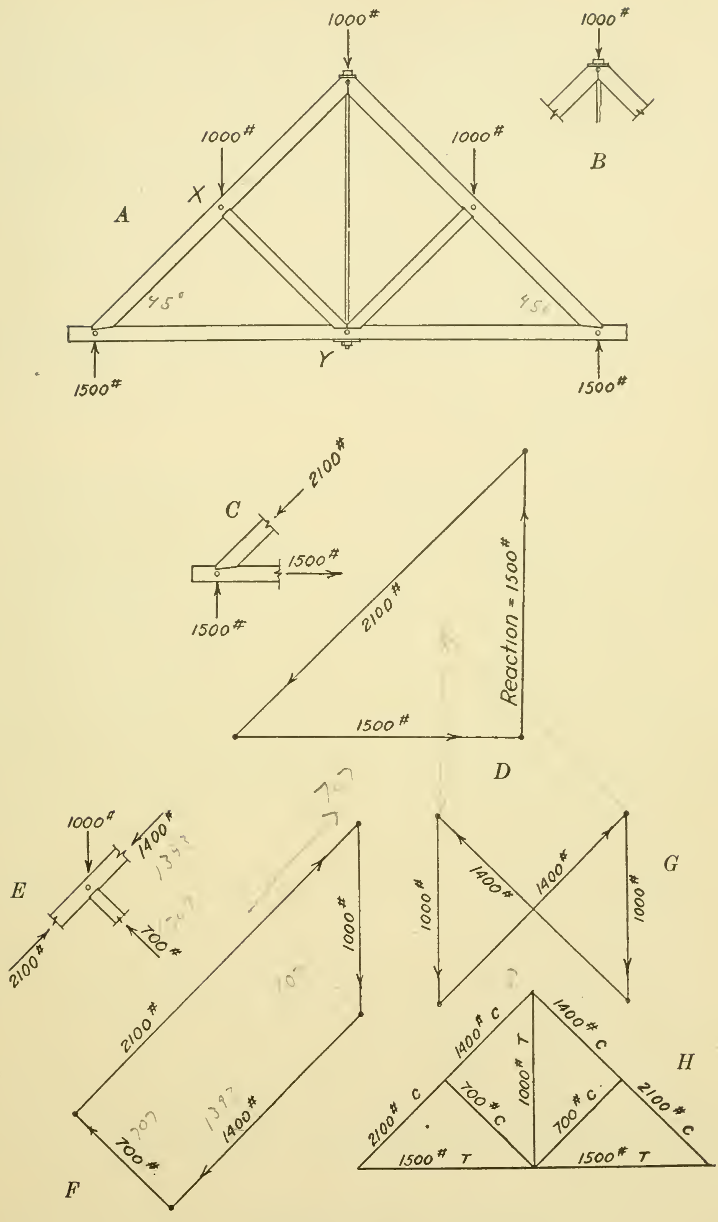

A
B
C
D
E
F
G
H
X
Y
1000#
1500#
2100#
1400#
700#
Reaction = 1500#
1400# C
2100# C
700# C
1000# T
1500# T

FIG. 34

external force is the reaction of 1500 lbs. acting upward. This force is made the initial line in the force diagram, Fig. 34*D*, in which the stresses in the two truss members are determined as 1500 lbs. tension and 2100 lbs. compression.

Figure 34*E* shows the next higher joint as a free body under the action of the given load of 1000 lbs. and the stress of 2100 lbs. compression just determined. These are used as the initial lines of the stress diagram, Fig. 34*F*, which in turn yields the stresses in two more members.

Passing next to the peak joint, we can find the stress in the vertical by means of a diagram like Fig. 34*G*. The stresses thus found are recorded on Fig. 34*H*. This is a space diagram and *not* a force diagram.

28. Summary. The preceding illustrations give a good idea of the application of the free-body method (§ 23) to the determination of the stresses in the various parts of a loaded structure. If these illustrations are fully understood and visualized, a good foundation will have been laid for future work. There are a few general considerations which are worth restating as a summary.

(*a*) When a framed structure is composed of straight members, so joined as to be free to turn at the joints, and when each load on the structure is applied at a joint, the members of the structure will be under simple axial stress. (Two-force pieces.)

(2) In such a case the stresses in the structure can be determined by the free-body method.

(3) Each joint in such a structure presents a typical problem in concurrent coplanar forces.

(4) The method for the solution consists in taking each joint in succession as a free body and solving for the stresses in the members which meet at that joint.

(5) The solution must be started at some joint where an external force (either a load or a reaction) is known, and where not more than two members are cut in taking the free body. (See (8) below.)

(6) In determining the *character* of the stresses at any joint, the clue is given by the direction of the external force.

(7) One joint having been solved, the stresses so determined become known forces acting on the succeeding joints.

(8) If there is no joint in the structure where one known force and only two unknown stresses occur, the stresses cannot be determined by the methods of statics. Such a structure is called statically indeterminate. (See § 40.)

PROBLEMS

NOTE. In the following problems, use a graphic solution. When the stress in a member has been found, it is convenient to record it on the corresponding line of the space diagram as shown in Fig. 37. To indicate character of stress use *C* for compression and *T* for tension.

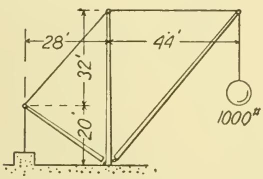


FIG. 35

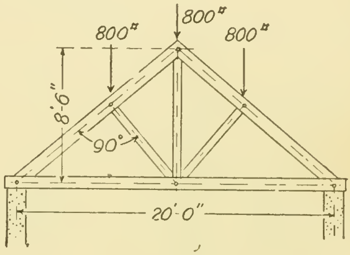


FIG. 36

1. Find the character and amount of the stresses in the members of the derrick shown in Fig. 35.
2. If the load on the derrick in problem 1 acts upward, and the amount of the load is 500 lbs., what will be the character and amount of the stresses?

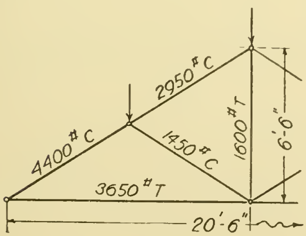


FIG. 37

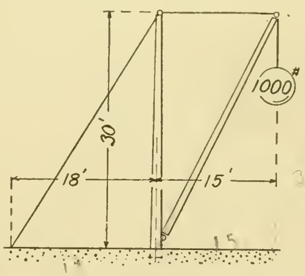


FIG. 38

3. If the load on the derrick in problem 1 acts downward and to the right, at 45° to the horizontal, what will be the character and amount of the stresses?
4. Find the stresses in the different members of the truss in Fig. 36 (scale 1″ = 400 lbs.).

5. If the span of the above truss is doubled, but the pitch and loading remain the same, what will be the stresses?
6. If the loading and span are as in problem 4, but the pitch is 45°, what will be the stresses?
7. If each load on the truss in problem 4 is doubled, what will be the stresses?
8. What loads (3 equal loads) will produce the stresses shown in Fig. 37?
9. Determine the stresses in the members of the derrick, Fig. 38.
10. What is the amount and character of the stress in each member of the frame in Fig. 39?
11. In problem 10, let the 500 lbs. force be applied at a, while still acting toward the right.
12. In problem 10, let a bar be inserted between b and c.

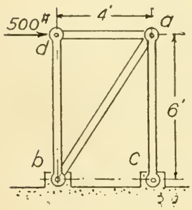


FIG. 39

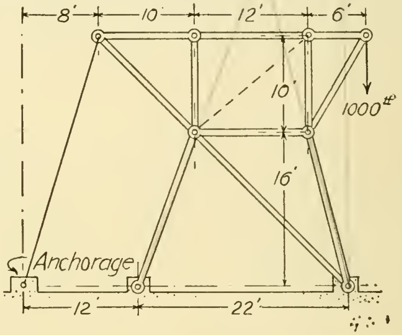


FIG. 40

13. Four pins are set in a wall at the corners of a 4′0″ square. The sides of the square are horizontal and vertical. A rope 20′0″ long is looped around and below the pins and carries a load of 1,000 lbs. suspended at its center. Determine the stresses in various parts of the rope and the resultant pressure on each pin. Neglect friction.
14. Determine the stresses in each member of the frame, Fig. 40.
15. In problem 14, remove the rod leading to the anchorage and insert a bar along the dotted line.

CHAPTER V

NON-CONCURRENT COPLANAR FORCES

29. Translation and Rotation. When a body moves so that each point of it changes its position by motion along a straight line, the motion of the entire body is one of ***translation.*** This is the kind of motion that has been frequently mentioned in the preceding chapters as resulting from the action of concurrent coplanar forces.

When a wheel turns on a fixed axle, the wheel *as a whole* does not move, but every point on the wheel is moving with respect to the axle. In this case the wheel is said to have a motion of ***rotation.*** Obviously, a body may have motions of translation and rotation simultaneously; as in the case of the wheels of a moving car.

In Fig. 41, we have a case where no motion of translation is possible (since $\Sigma H = 0$ and $\Sigma V = 0$) but evidently there will be a motion of rotation. We must then look for some new condition of equilibrium to govern such cases. The conditions of equilibrium for concurrent coplanar forces (§ 17) take into account the amounts and directions of the forces concerned, but they are not concerned with the position of the forces, since all forces in such a problem have a common point of application.

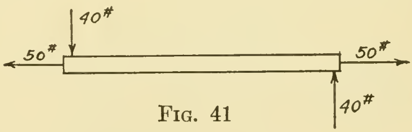


Fig. 41

We may therefore expect that for non-concurrent forces the necessary third condition for equilibrium will have to do with the position (point of application) of the forces.

Most of the problems that occur in connection with structures involve forces in a single plane. Those that involve forces in more than one plane can usually be solved as a series of coplanar solutions. Therefore all the problems of this chapter deal with coplanar forces only. Some mention of non-coplanar forces is made in Chapter XXIV.

30. Moment of a Force. Figure 42A shows a steelyard, which illustrates the principles sought. The heavy weight X hangs nearer the support and is balanced by a smaller one Y at a greater distance. In Fig. 42B the beam of the steelyard is shown as a free body. Unless the vertical force through the support is equal to $X + Y$, motion of translation, in a vertical sense, will occur. Moreover, equilibrium can exist when and only when $Xb = Ya$; i.e., when one force times its lever arm is equal to the other force times its lever arm. This is the well-known law of the lever. It is so much a matter of common experience and can be verified so easily by simple experiments that it requires no proof.

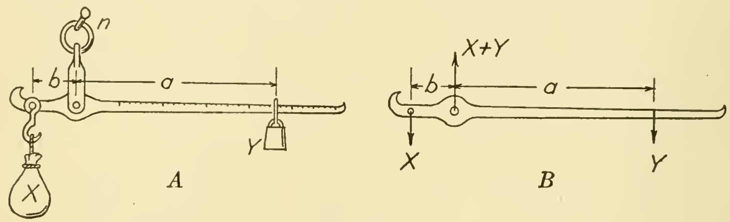


FIG. 42

The result of multiplying a force by a distance is a compound quantity called the ***moment of the force.*** Moments are expressed in the same terms as their component parts (pound-feet, pound-inches, ton-inches, etc.).

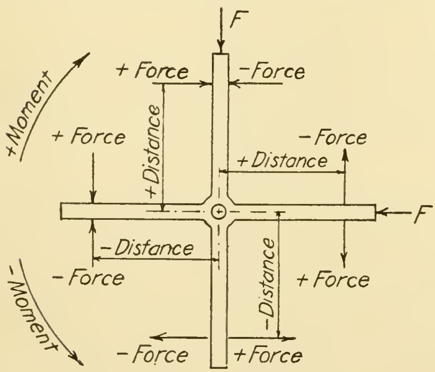


FIG. 43

A moment may be positive or negative. Those producing (or tending to produce) rotation in a clockwise direction about a center (as Ya, Fig. 42B) are called positive.* Those producing counter-clockwise rotation are called

* Figure 43 shows the conventional system of signs for force and distance. The sign of the resulting moment is given by combining the signs of its component force and distance as in algebraic multiplication.

negative. When the lever arm of the force is zero, i.e., when its line of action passes through the center of rotation, its moment is of course zero. (See F, Fig. 43.)

31. Lever Arm. The example used in § 30 is a special case, since all forces are vertical. In order to establish a perfectly general definition for a moment let us take such a case as shown in Fig. 44.

It is evident that the tendency of the force X to produce rotation about o is less than it would be if the same force were vertical. In order to determine the true moment, let X be resolved into its H and V components. Since these components are, by definition, forces that produce the same effect as the original force, the total moment of the components about the center o must be the same as the moment of X about the same center.

The components of X are $X \sin \theta$ and $X \cos \theta$, as shown; and the lever arms of the components are b and zero respectively. Hence the total moment of rotation due to the two components is $(X \sin \theta)b + (X \cos \theta)0 = (X \sin \theta)b$. Now if a perpendicular is dropped from o to the line of action of X, its length will be $b \sin \theta$. If this perpendicular is used as the lever arm, the moment of X about o is $X \sin \theta\, b$, which agrees with the moment of the components. Therefore we may say that *the moment of any force is found by multiplying the force by the perpendicular distance from its line of action to the center of rotation.*

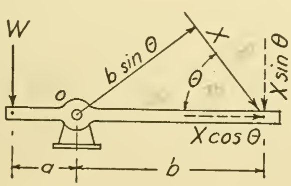


FIG. 44

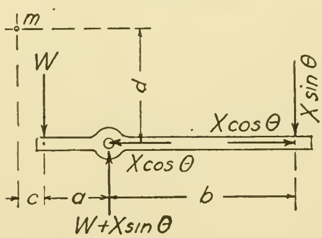


FIG. 45

32. Conditions of Equilibrium. Figure 45 shows the beam and loadings of Fig. 44, as a free body. Let it be assumed that W is a force that will keep the beam in equilibrium. Also let the reaction of the support and the force X be replaced by arrows

representing their H and V components. For equilibrium of translation there must be a vertical upward reaction of $W + X \sin\theta$ and a horizontal, leftward reaction of $X \cos\theta$. The center of rotation is not now so easy to locate, but a little reflection will show that if the body is in equilibrium there is no tendency to rotate about *any* point. Hence if *any* point is chosen as a center of rotation (or better as a *center of moments*), the resulting summation of moments must give zero.

For example, let the point m (any point) be chosen as a center of moments. The moment equation now becomes:

$$\Sigma \text{ moments (about } m) = (Wc) + (X \sin\theta)(a + b + c) \\ + (X \cos\theta)d - (W + X \sin\theta)(a + c) - (X \cos\theta)d.$$

Simplifying this expression, we find

$$\Sigma \text{ moments (about } m) = Xb \sin\theta - Wa.$$

From the principles of §§ 30–31, $Wa = Xb \sin\theta$. Hence we have

$$\Sigma \text{ moments (about } m) = 0.$$

That is to say, if any system of forces is in equilibrium, the sum of the moments of *all* the forces, using *any* point in the plane of forces, as a center of moments, is zero.*

This establishes the third condition for equilibrium of non-concurrent coplanar forces, forecast in § 29. When this condition is satisfied, no motion of rotation can occur. Likewise, if no motion of translation is to occur, the conditions of equilibrium for that case (§ 17) must *also* be fulfilled.

We summarize what precedes in the statement that if non-concurrent coplanar forces are in equilibrium each of the equations

$$\Sigma H = 0, \qquad \Sigma V = 0, \qquad \Sigma M = 0$$

must be satisfied *separately*.

In § 19 it was shown that in the case of concurrent forces two and only two unknown quantities can be determined from the two conditions of equilibrium which apply to that case.

* Let the student draw a figure similar to Fig. 45 and prove the above proposition without replacing the force X by its components.

In the same way, in the case of non-concurrent forces, the three conditions of equilibrium lead to three equations from which three and only three unknowns may be determined.

Sometimes a solution is possible when there are apparently more than three unknown quantities. But in such a case there is some condition unrelated to the laws of equilibrium which may be used to establish the necessary equations (§ 35).

33. Reactions. By means of the preceding laws of equilibrium we now may determine the reactions due to the loading of a structure or of any structural part. Figure 46 shows a ship's crane, carrying a load L. Let it be required to find the supporting forces (reactions), which must be furnished by the two decks.

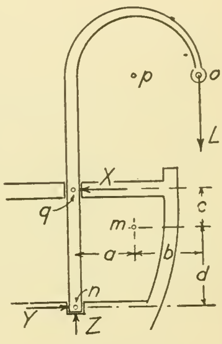


FIG. 46

The upper deck, because of the way it is constructed, can furnish only horizontal support; hence all the vertical reaction must be furnished by the lower deck. It is evident that the upper and lower decks must each furnish a horizontal reaction to prevent rotation. Let these three unknowns be X, Y, and Z as shown in the figure. Then we have

$$\Sigma V = 0, \qquad \text{or} \qquad L + Z = 0,$$
$$\Sigma H = 0, \qquad \text{or} \qquad Y + X = 0,$$

and, taking moments about m as a center,

$$\Sigma M = 0, \qquad \text{or} \qquad Lb + Za - Xc - Yd = 0.$$

Solving these equations, we find

$$Z = -L,$$
$$Y = -X,$$
$$X = \frac{L(a+b)}{(c+d)}.$$

These results give the amounts of the unknown reactions in terms of the known load and known distances.

34. Choice of a Center of Moments. By a judicious choice of a center of moments, the algebraic work often may be materially shortened. In the above example, if the center had been taken at n two forces (Z and Y) would have passed through the center of moments. The moments of these forces would then be zero, and they would disappear from the equation. Then the equation could be written immediately in the form

$$L(a + b) - X(c + d) = 0,$$

from which we find at once the value of X.

$$X = \frac{L(a + b)}{(c + d)} \cdot$$

In order to test out this matter let the student assign a definite value for L, and definite values for the various distances, and then try using the points n, o, p, and q as centers of moments.

After working this problem through, it will be seen that the value of Z might have been stated by inspection, since Z and L are the only V forces in the problem.

PROBLEMS

1. In Fig. 42, if Y weighs 14 oz., the distance $a = 36''$ and $b = 2\frac{1}{2}''$, what is the weight of X and what is the pull on the nail n?
2. In Fig. 42, if $a = 36''$, $b = 5''$, and the nail n can support 150 lbs., how great may be the weights X and Y?
3. In Fig. 44, if X is 100 lbs. and θ is 30° while $a = 12''$ and $b = 36''$, how large must W be to maintain equilibrium?
4. In problem 3, what will be the amount and direction of the reaction at o?
5. Find the amount, direction, and point of application for a single force that will hold the bar in Fig. 47 in equilibrium.
6. Find the amount and direction of the reaction in Fig. 48. Take a as 6′0″.
7. What would be the result in problem 6 if $a = 4.09$ ft.?

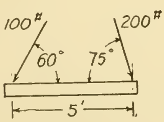


FIG. 47

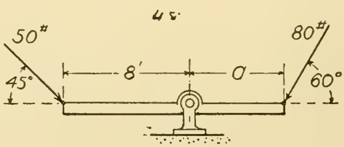


FIG. 48

35. Reactions of a Derrick. Another problem in the determination of reactions by means of moments, and one which varies somewhat from the former in detail, is shown in Fig. 49.

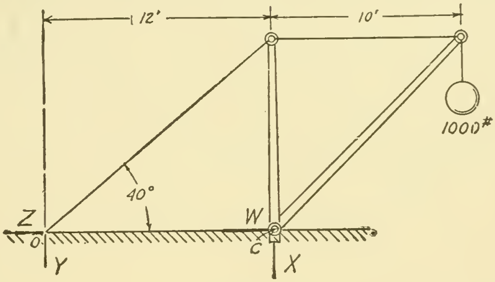


FIG. 49

Let it be required to find the reactions at the foot of the mast and back-stay. Evidently each reaction will have a V and an H component, as shown by W, X, Y, and Z. None of the values can be stated by inspection, and, since the problem involves four unknown quantities, it could not be solved (§ 32) if it were not for the fact that the direction of the resultant of Z and Y is known to coincide with the back-stay. This fact will yield the necessary fourth equation. Hence we can now state the equations

$$X + Y + 1000 = 0 \qquad (\Sigma V),$$
$$Z + W = 0 \qquad (\Sigma H),$$
$$12X + 22000 = 0 \qquad (\Sigma M) \quad \text{(about } o \text{ as a center)},$$
$$\frac{Y}{Z} = \tan 40°.$$

Solving these equations, we find

$$X = -1833, \qquad Z = -992,$$
$$Y = 833, \qquad W = 992.$$

The signs of these values indicate the directions for the arrows in Fig. 50.

It is often possible to simplify such a solution by determining the signs of the forces by inspection. Thus, in Fig. 49, it is easy to see that the load tends to tip the derrick downward and to

the right. Therefore the pull at o must be downward and to the left and the thrust at c must be upward and to the right.

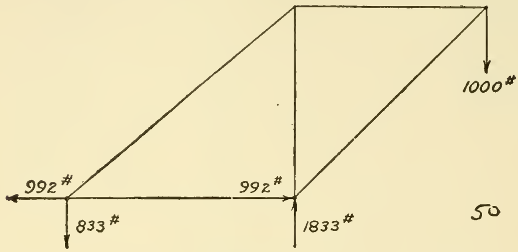


FIG. 50

From these considerations we have,

$$\text{taking a center at } o\text{: } 12X = 22000,$$

$$\text{" " " " } c\text{: } 12Y = 10000,$$

$$\frac{Y}{Z} = \tan 40°.$$

From these equations, $X = 1833$; $Y = 833$; $Z = 992$, and (since $Z = W$) $W = 992$. Each of these results is positive in *the direction assumed.* Had one or more of these results been negative, it would have indicated that the sense assumed for that force was incorrect.

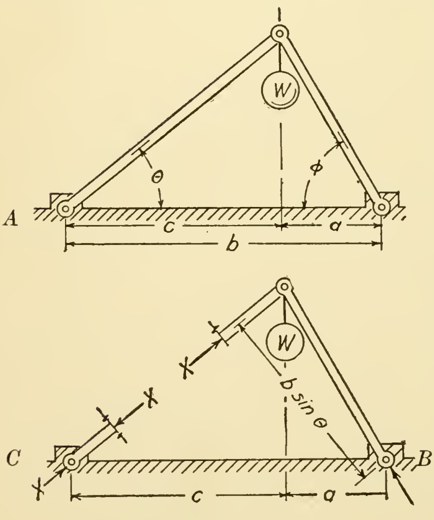


FIG. 51

We might have solved this problem in a still different manner by starting from the end of the boom, solving for the stresses in boom and top-stay, and then passing on to the top of the mast and down to the reactions after the manner of § 25.

36. Stresses determined by means of Moments. The moment equation is sometimes very useful in finding stresses as well as reactions. Thus in Fig. 51*A*, let it be required to find the internal stresses in the bars as well as the reactions of the supports, all dimensions being known. Take the right-hand portion as a free body (Fig. 51*B*), substituting an arrow for the stress in the left-hand member. Now taking the right-hand pin as a center, we see that the moment of the right-hand reaction will be zero, and that the moment equation may be written in the form

$$Xb \sin \theta - Wa = 0,$$

whence we find

$$X = \frac{Wa}{b \sin \theta}.$$

The left reaction is found to be equal to X since the bar is a two-force piece. (See Fig. 51*C*.) Let the student now find the stress in the right-hand bar.

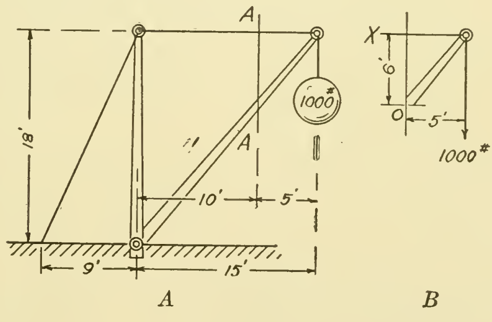


FIG. 52

Another case illustrating the same method is shown in Fig. 52*A* (compare with Fig. 49). Let it be required to find the stresses in the boom and top-stay, using the method of moments. Pass a cutting plane (*AA*), so that it cuts both the members at some known convenient section. Take the portion to the right of the plane as a free body (Fig. 52*B*). Now taking a center of moments at *o*, we find that the moment of the load is 1000 lbs. × 5′ = 5000 lbs.-ft. (clockwise). The (unknown) stress x in the top-stay must produce an equal and opposite moment.

Hence we have

$$6x = 5000, \qquad \text{or} \qquad x = 833.$$

Let the student determine to what extent these computations would have been changed if another cutting plane had been used. Also determine the stresses in the back-stay and in the vertical mast by the same method.

37. Transmissibility of Force. The ball shown in Fig. 21 is supported by the upward pull in the cord balancing the force of gravity. So long as this pull acts in the same line with the force of gravity, it does not matter (so far as the equilibrium of the ball is concerned) at what point on the cord the supporting nail is placed, nor to what part of the ball (top, center, or bottom) the cord is attached. For that matter it would amount to the same thing if the support were furnished from below, as by a post. The necessary condition is that the supporting force act in the same line with the force of gravity, but it is immaterial at just what point in that line the supporting force acts.

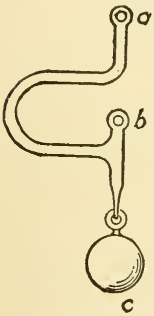


Fig. 53

Again, in Fig. 53, the body might be hung from a or b or from any point on the line ac, and it would take exactly the same position. That is, in so far as equilibrium alone is concerned, it does not matter at what point on the line of action of the applied force the support occurs. Of course the stresses in the supporting bar will vary, but for the present we are concerned only with external forces.

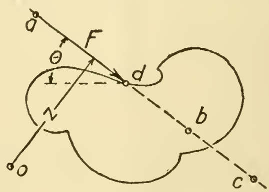


Fig. 54

In order to generalize these facts, let Fig. 54 represent any body acted upon at the point d, by any force F, which is inclined at any angle θ to the horizontal. Let o be any point chosen as a center of moments. The effect of F in producing motion of translation will be measured by its components, $F \sin \theta$ (vertical)

and $F \cos \theta$ (horizontal), and its effect in producing rotation about o is measured by its moment, Fz.

Now let F be moved from d to b. It is evident that neither its components nor its moment about o is changed in the slightest; hence the motion produced will be the same as before. Moreover, if the point c or a, or any other point in the line ac and outside the body, be considered as rigidly connected to the body, the force might be transferred to that point and still produce the same kind of motion as in the first instance.

This discussion justifies the statement that *any force acting on a body produces the same effect no matter where, in its line of action, it is considered as applied to the body or the body extended.* This principle is known as the principle of transmissibility of force.

PROBLEMS

1. Determine the stresses and reactions in Fig. 55, using the method of moments.
2. How will the stresses and reactions of problem 1 vary:
   (*a*) if the ring takes successively the positions 1 to 6.

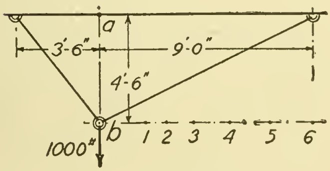


FIG. 55

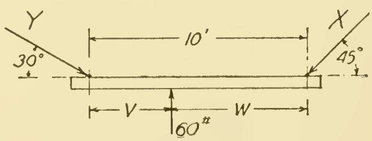


FIG. 56

   (*b*) if the ring stays on the line ab while the cords are lengthened.
   (*c*) if the ring stays on the line ab while the cords are shortened.
3. Find the distances V and W and the amounts X and Y so that equilibrium may exist in Fig. 56.

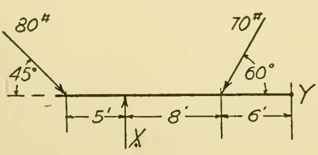


FIG. 57

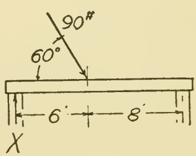


FIG. 58

4. In problem 3 change the 60 lbs. force to a downward force and solve as before.

5. In Fig. 57: A force of 30 lbs. (direction unknown) acts at Y. A force, Z, of 20 lbs. acts vertically upward (point of application unknown). The amount of X is unknown. Determine these three unknowns so that the bar is in equilibrium. (A solution partly analytic and partly graphic is suggested).
6. Determine the reactions in Fig. 58.

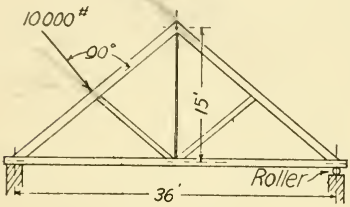


FIG. 59

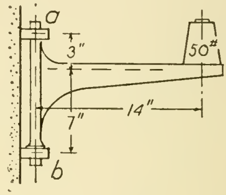
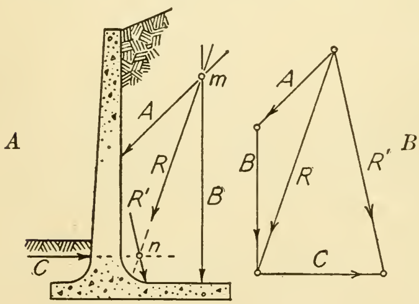


FIG. 60

7. Determine the reactions in Fig. 59 (analytic solution).
8. Determine the reactions for the derrick in Fig. 38 by the method of moments. Also determine the stresses in the members by the same means.
9. Determine the reactions and stresses for the derrick shown in Fig. 35, using method of moments.
10. Find the direction and amount of the reactions at a and b in Fig. 60.

38. Resultant. If any system of non-concurrent coplanar forces is not in equilibrium, the resultant may be found by means of the principle of § 37. In Fig. 61A is shown a retaining wall acted upon by the forces A, B, and C (non-concurrent). A and B are concurrent and have a resultant equal to R, Fig. 61B, which must act through m, Fig. 61A.



Space diagram. *Force diagram.*

FIG. 61

The forces R and C can now be combined into the resultant R', which must act through n. Notice that the direction and amount of the resultant is found on the force diagram, while its point of application is found on the space diagram.

Since any system of non-concurrent coplanar forces can thus

be reduced to separate systems of concurrent forces and the resultants of these can be combined again, the preceding method will suffice for all problems of this nature unless the forces themselves or their resultants are parallel. This case will be taken up later.

NOTE. The preceding articles give the general principles for the solution of problems involving non-concurrent coplanar forces. The articles which follow will take up some important special cases.

39. Equilibrium of Three Forces. Three forces cannot be in equilibrium unless they are concurrent. For any two of them would have a resultant (as R, Fig. 61). This resultant and the third force will also have a resultant, R', unless R is equal and opposite to the third force and acts in the same line with it. In the latter case the three original forces are concurrent.*

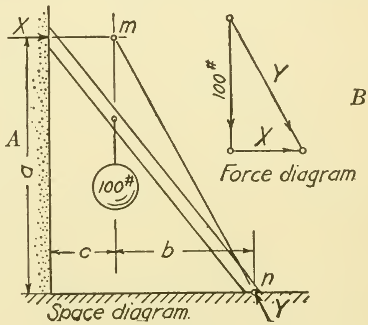


FIG. 62

This principle can frequently be used to advantage, as in the following example. In Fig. 62, let a beam be considered as resting on the ground and against a frictionless† vertical wall, and carrying the load shown. Let it be required to determine the amount of X and the amount and direction of Y.

Here we have a body acted upon by three forces. The direction of the load and of X being known, their point of intersection m can be found. Since the three forces must pass through one point, that point must be m and therefore Y must act in the direction nm. Now let a force diagram be laid out to scale, as at B. The 100 lbs. load is first drawn. Then a line to represent

* An apparent exception to this principle is in the case of three parallel forces. If, however, parallel lines are thought of as meeting at an infinite distance, the exception disappears.

† Friction is the resistance to the motion of one surface over another with which it is in contact. (See also § 241.) It is, therefore, a force whose direction is the same as that of the surface in which it occurs. Hence, when surfaces are frictionless, the pressure between them must of necessity be normal to the surfaces. In this problem this consideration fixes the direction of the force X as horizontal.

X, horizontally. Finally the triangle is closed by a line parallel to mn, Fig. 62A. The amounts of the forces X and Y can now be scaled from Fig. 62B. This problem can also be solved analytically, by the method of moments. Let the student prepare such a solution.

40. Statically Indeterminate Cases. In § 19 we have seen that in the case of concurrent forces not more than two unknown quantities can be determined from the laws of statics. Similarly in § 32 we found that not more than three unknowns can be determined in a case of non-concurrent forces.

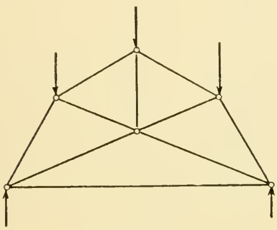


Fig. 63

There are cases which arise constantly where there are three, four, or more unknowns. Such cases are statically indeterminate. That is, they cannot be solved by means of the laws of statics. The following examples will illustrate the point.

A. Concurrent Forces. In Fig. 63 is shown the diagram of a roof truss. No matter what joint is taken as a free body, at least three members with unknown stresses are found. Hence, no statical solution is possible since the problem is one of concurrent forces. (Compare with § 19.)

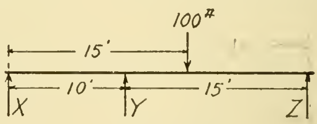


Fig. 64

B. Non-concurrent Forces. These cases occur in two typical forms.

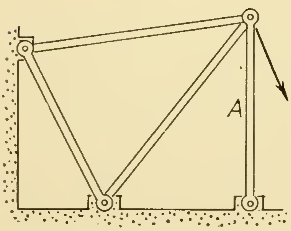


Fig. 65

(1) *Wholly indeterminate cases.* In Fig. 64 is shown a beam resting on three supports. There are only two equations for the determination of the reactions, since no horizontal forces are present. Therefore the problem is indeterminate.

In Fig. 65 is a case in which the reactions may have both

horizontal and vertical components, and each component is an unknown; therefore the reactions are indeterminate. Similarly the internal stresses are indeterminate, since no joint exists at which one known and not over two unknown forces are present. If the strut A is removed, the problems of reactions and stresses become determinate.

(2) *Partially determinate cases.* In Fig. 66 is shown a roof truss, acted upon by the wind load W. Let it be required to find

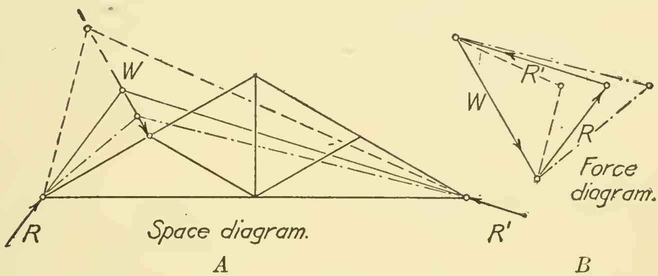


Fig. 66

the reactions. Each reaction is unknown as to its amount and direction. There are thus four unknown quantities, as in the previous case. The truss is a body acted upon by three forces, R, R', and W. These three forces must meet at a point (§ 39). One possible solution based on this fact is shown by the solid

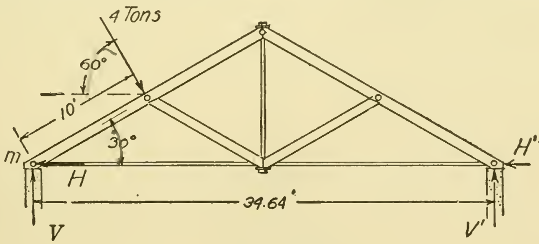


Fig. 67

lines on the space and force diagrams. But other solutions, as shown by the dotted and broken lines, are possible. There is nothing to determine which solution is the correct one. But it will be noticed that, on the force diagram, all of the lines representing reactions intersect on the same horizontal line. This

means that the V components of the reactions are the same for any possible solution and are determinate although the H components are indeterminate.

An analytic solution of the same case shows the same thing. Let the student, using the data of Fig. 67, compute the V components of the reactions and show that the H components are indeterminate, by an analytic process.

C. Other Solutions. Some statically indeterminate cases can be solved by methods involving the stiffness of materials and other more advanced principles. In some cases reasonably accurate solutions may be found by making certain reasonable assumptions, as in § 261.

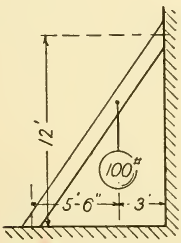


Fig. 68

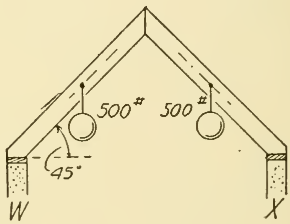


Fig. 69

But any problem involving more than two (in the case of concurrent forces) or three (in the case of non-concurrent forces) unknown quantities, whether these be amounts, directions, or points of application, is statically indeterminate.

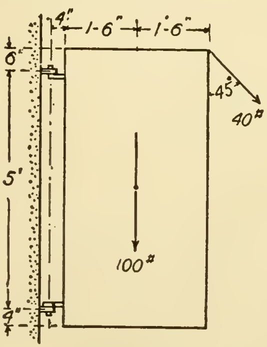


Fig. 70

PROBLEMS

1. Find the stress in the lower chord of the truss in Fig. 36 by the method of moments.
2. Suppose the wall in Fig. 68 is frictionless. How much friction must be developed on the floor to support the load?
3. In problem 2 let the friction possible to be developed on the floor be 30 lbs. What weight may be carried?
4. What is the thrust on the walls in Fig. 69?
5. Find the reactions of the hinges in Fig. 70. Check by graphic method.

6. In Fig. 62 assume the wall to exert friction. Solve for the reactions.
7. In Fig. 67 assume the right-hand wall to be frictionless. Solve for the reactions (analytic solution). Check results graphically by a diagram, similar to Fig. 66.
8. Draw a diagram illustrating a statically indeterminate case.

41. Parallel Forces. In the study of concurrent forces (§ 19 and footnote), we saw that since one of the force characteristics (point of application) is common to all the forces concerned, only two conditions of equilibrium can be established; but that these are sufficient to determine any two unknown quantities.

In the case of parallel forces much the same condition exists. All the forces have a common direction. Hence no component perpendicular to this direction can exist, either in the forces themselves or in the resultant or equilibrant of the forces. Therefore only two equations can be set up, one based on the summation of amounts of the forces and the other on their moments. This means of course that only two unknown quantities can be determined.

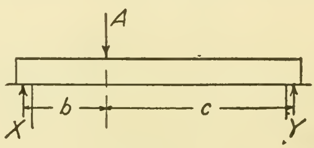


FIG. 71

The case of a beam carrying a vertical load (Fig. 71) will illustrate the matter. Since the load is vertical there is nothing to set up H components in the reactions. Therefore the reactions will be vertical but of unknown amounts. If A, b, and c are known, we have the equations

$$0 = 0, \qquad (\Sigma H = 0), \tag{1}$$

$$A + X + Y = 0, \qquad (\Sigma V = 0), \tag{2}$$

and, taking moments about X,

$$Ab + Y(b + c) = 0, \qquad (\Sigma M = 0). \tag{3}$$

Solving (3) for Y, we get

$$Y = \frac{-Ab}{b + c}.$$

Substituting this in (2), we find

$$X = -A\left(1 - \frac{b}{b + c}\right).^*$$

In the preceding case the amounts of the two reactions were found. The problem may occur in other forms, as indicated in Fig. 72, where X and y represent the unknown quantities. In each case the above method will suffice.

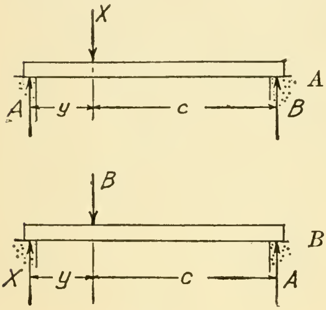


FIG. 72

The introduction of more known forces and distances changes neither the principle involved nor the method of solution. Thus, in Fig. 73, taking moments about X, we may find the value of X and Y from the equations

$$X + Y = 100 + 200 + 400,$$
$$20Y = 1200 + 6000 - 400.$$

Solving these equations, we find $Y = 340$, $X = 360$.

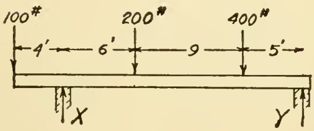


FIG. 73

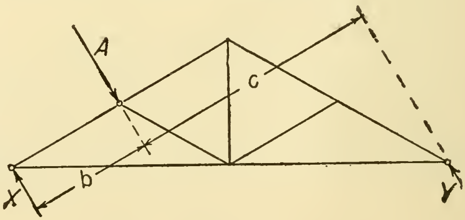


FIG. 74

When parallel forces that are not actually vertical are involved, the same principles hold true. Figure 74 shows a roof truss acted upon by the wind. It is commonly assumed† that the

* Very often the sense (§ 9) of the unknown forces can be determined by inspection. Thus, in Fig. 71, we can say that X and Y must act in the opposite sense to A. It is then immaterial whether we write the equation in the form $X + Y + A = 0$ or in the form $A = -X - Y$, or $A = X + Y$, provided only that in the latter case we realize that it is the arrows on the figure (which have already been determined by inspection) which really indicate the sense of forces. When the sense of the unknown cannot be determined by inspection, the shorter form cannot be used. Hereafter either form may be used in the text.

† It will be remembered (§ 40) that this case is statically indeterminate. When it arises in practice, however, this assumption is commonly made as being a reasonably close approximation (§ 261).

reactions in such a case are parallel to the load. Then we have

$$(1) \qquad X + Y = A,$$

$$(2) \qquad Y(c + b) = Ab.$$

Solving these equations we find

$$X = \frac{Ac}{c + b}, \qquad Y = \frac{Ab}{c + b},$$

and therefore

$$\frac{X}{Y} = \frac{c}{b}.$$

That is to say, the reactions are inversely proportional to their distances from the load. This principle often makes it possible to solve such a problem by inspection. (See Fig. 75.)

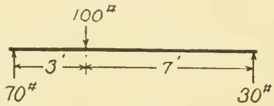


Fig. 75

42. Reactions of a Beam by Composition. In determining beam reactions it is sometimes advantageous for us to treat each load separately, determining the reactions due to each alone and then combining the results by addition. Thus in Fig. 76 the reactions due to each load separately can be determined by inspection. The total reactions are found by simple addition of the component parts. The case shown in Fig. 77 furnishes an illustration of a more complex case in which the load can be divided into (*a*) a uniformly distributed load of 100 lbs., (*b*) a uniformly increasing load whose total amount is 150 lbs., and (*c*) a concentrated load. The component and total reactions can be determined by inspection as

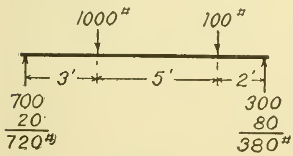


Fig. 76

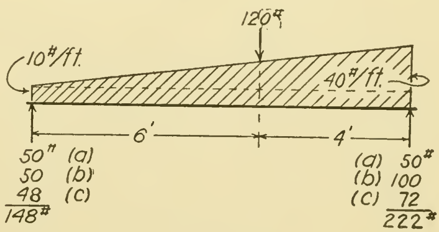


Fig. 77

shown in the drawing. Of course such a solution cannot always be used to advantage. It will depend on the amounts of the loads and distances involved as well as the total number of quantities in the problem. Occasionally such a method is useful as a check on a computation performed in a different manner.

43. Resultant of Parallel Forces. The principles of § 41 not only give the solution for cases where two reactions are to be found but also they can be applied to finding a single reaction due to two or more forces or to any problem of a similar nature.

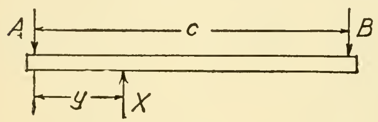


Fig. 78

Applying the above principles, the conditions in Fig. 78 give, for equilibrium, the equations

$$X = A + B, \qquad (\Sigma V = 0),$$
$$Xy = Bc, \qquad (\Sigma M = 0, \text{ center at } A).$$

The solution of these equations will give the reaction X in amount and position. (It will be noticed that this figure is the same as Fig. 71 turned upside down.) Here we have determined the amount and position of the anti-resultant of A and B. Now the resultant of A and B must be equal and opposite to, and act in the same line as, X.

Hence, to find the resultant of parallel forces, we need only use the preceding principles, remembering that the resultant acts in a direction opposite to that of the anti-resultant.

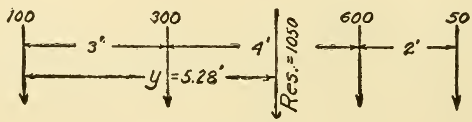


Fig. 79

Thus to find the resultant of the forces in Fig. 79, we have

$$\text{Resultant} = 100 + 300 + 600 + 50 = 1050, \qquad (\Sigma V = 0),$$

and, taking moments about the left hand force,

$$1050y = (100 \times 0) + (300 \times 3) + (600 \times 7) + (50 \times 9), \qquad (\Sigma M = 0);$$

whence $y = 5.28$.

44. General Expression for Position of Resultant. In the discussion of center of gravity, it will be convenient to have a general expression for the position of the resultant of parallel forces. Moreover it will tend to fix the ideas of § 41 if they are made perfectly general.

For this purpose let A, B, C, etc., be any set of parallel forces and a, b, c, etc., be the distances of these forces from some (arbitrary) center of moments. Now the amount of the resultant of these forces is evidently $A + B + C +$ etc. Let z be the (unknown) distance from the center of moments to the resultant. Then the moment of the resultant is equal to the sum of the moments of the forces, i.e.,*

$$(A + B + C + \text{etc.})z = Aa + Bb + Cc + \text{etc.},$$

$$z = \frac{Aa + Bb + Cc + \text{etc.}}{A + B + C + \text{etc.}},$$

which may be written in the form†

$$z = \frac{\Sigma(\text{force} \times \text{distance})}{\Sigma\ \text{force}}.$$

* Notice that the value of z thus found is necessarily expressed in terms of distance since the fraction is in the form of $\frac{\text{distance} \times \text{force}}{\text{force}}$.

† Most students are inclined to regard such an expression as a formula to be retained by a sheer feat of memory. If, on the other hand, one should come to regard it as a condensed representation of the perfectly logical and necessary relations which exist between a certain set of physical quantities, he would eliminate at once both the tiresome effort and the large chance of error that always accompanies a pure feat of memory.

For example, in the above expression, z represents a sort of weighted average distance of forces, measured from an arbitrary center. Obviously any such quantity must be affected by the amount and the position of *every* force in the problem. Moreover the *amount* of the resultant is influenced only by the *amounts* of the forces. Again the *moment* of the resultant must be the same as the *moment* of the forces, about any point. That is,

$$z(\Sigma\ \text{force}) = \Sigma(\text{force} \times \text{distance})$$

which is identical with the expression above. After a little study of such an expression one can get to "feel" both the reasons behind it and its inevitableness. Then forgetting becomes impossible.

45. Structure Containing Multi-force Pieces. The structures so far noticed are all composed of two-force pieces. (Footnote, page 27.) In Fig. 80 is shown a derrick in which the members *A* and *B* are two-force pieces while *C* is a three-force piece and *D* is acted upon by four forces.

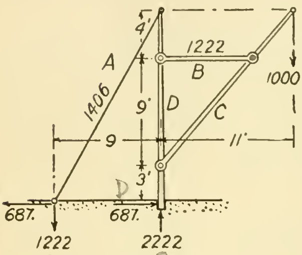


FIG. 80

In such a case the stresses in the two-force pieces can be determined by moments as in § 36. The reactions can be determined by the same means (§ 35). Let the student check the amounts of the stresses and reactions shown in the drawing.

The forces acting on *C* and *D* are, in whole or in part, directed transversely to the axis of the member. Such forces produce bending, a phenomenon which is much more complex than tension or compression and which is discussed in Chapters XIII to XXII.

PROBLEMS

1. A beam 12′ long carries a load of 1440 lbs. at 3′ 8″ from the left end. Determine the reactions.
2. In Fig. 72*A* and *B*, find *X* and *Y* in terms of *A*, *B*, and *c*.

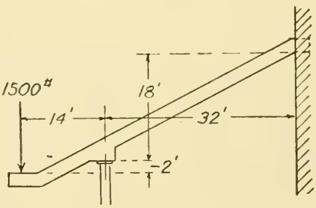


FIG. 81

3. Find the reactions in Fig. 81. Neglect weight of beam.

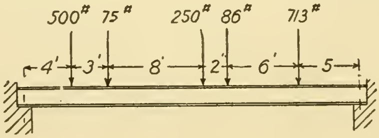


FIG. 82

4. Find the reactions in Fig. 82. Neglect weight of beam.

5. In Fig. 73, evidently each load has its influence in determining the reactions. Compute the reactions due to each load as if acting alone. Add these and check with results on page 54.

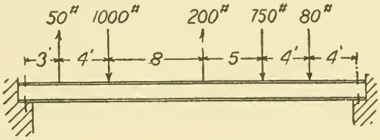


Fig. 83

6. Determine the reactions in Fig. 83. Neglect weight of beam.
7. Determine the reactions in Fig. 84. Neglect weight of beam.
8. Determine the reactions in Fig. 85. Neglect weight of beam.
9. Determine the resultant in Fig. 86.

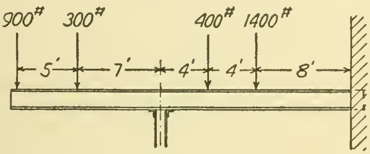


Fig. 84

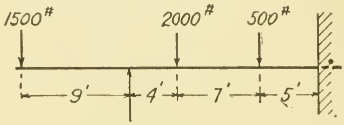


Fig. 85

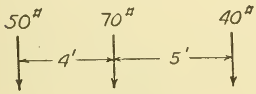


Fig. 86

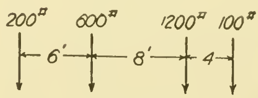


Fig. 87

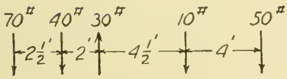


Fig. 88

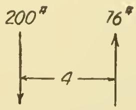


Fig. 89

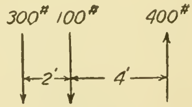


Fig. 90

10. Determine the resultant in Fig. 87. Compare with Fig. 79.
11. Find the resultant in Fig. 88.
12. Find the resultant in Fig. 90.
13. Find the equilibrant in Fig. 89.
14. What forces X and Y, Fig. 91, will balance the given load?

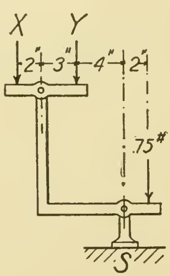


Fig. 91

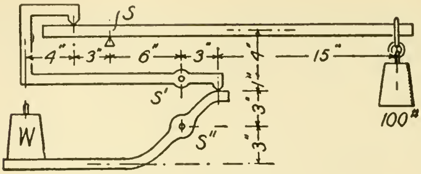


FIG. 92

15. What weight W, Fig. 92, can be supported by the given load?
16. Find the amount and direction of the reaction in Fig. 93. Also the stress in the stay rod A.
17. Determine the reactions and the stress in each two-force piece in Fig. 94.

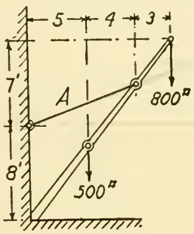


FIG. 93

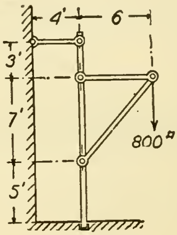


FIG. 94

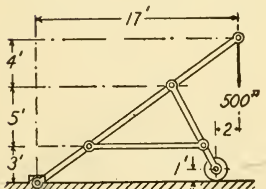


FIG. 95

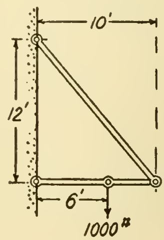


FIG. 96

18. Repeat problem 17, using Fig. 95.
19. Repeat problem 17, using Fig. 96.

CHAPTER VI

CENTER OF GRAVITY

46. Physical Significance. In the problems thus far studied, the various bodies and structures used have been considered as weightless, though it is evident that in any real case the results would be affected more or less by the weights thus neglected

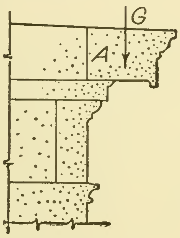


Fig. 97

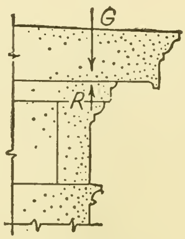


Fig. 98

and that in many cases the weight of a structure itself may be of prime importance in determining the actual stresses which are present. It will be the purpose of this chapter to develop principles by the use of which the force of gravity acting on a body or the parts of a structure may be expressed by an arrow or arrows in the same way as has been done heretofore in the case of the arbitrarily selected forces which have been used.

Figures 97 and 98 show sections through two wall copings. We instinctively recognize that the stone A, Fig. 97, is in a precarious position, while in Fig. 98 the corresponding stone is securely balanced on the wall. While we recognize these facts quite readily, it is not always so easy to see clearly the reasons for them. Each particle of the stone is acted upon by a force (gravity) and all such forces acting on a given stone constitute a set of parallel forces which will of course have a resultant. This resultant may be expressed as a single vertical force G. When the position of this resultant is such that a reaction R

(Fig. 98) can be offered by the lower courses, the stone may be supported on the wall.

It will be noticed that in the above discussion the sole matter of importance is the *position* of the resultant force. The conclusions arrived at would be the same whether the force G be large or small. Furthermore the position of G would be relatively the same if the stones were made either larger or smaller while keeping the same proportions. It follows then that the matter of prime importance in locating G is the *distribution* of the weight of the stone.

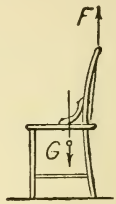


Fig. 99

The same principle is encountered when one raises a chair. If we lift straight up on the back as indicated by F, Fig. 99, the resultant force of gravity G will cause the chair to rotate until it assumes the position shown in Fig. 100. Here the forces F and G act in the same straight line and can produce equilibrium. In this case if the *position* of the force G can be determined, it will be possible to foretell just how the chair will hang at rest when suspended from any given point.

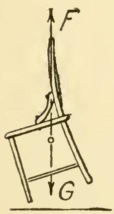


Fig. 100

Wherever the problem of balance is encountered, whether it be in a juggler's trick or in the arrangements for hoisting bulky loads, this question of the position of the resultant force of gravity* is the essence of the problem.

Since the problem outlined above is one of position and since position is a space relation, it follows that the whole matter has a strong geometric character. The matter of symmetry about one, two, or three planes becomes of prime importance and, in general, the geometric properties of the bodies dealt with often contain the key to the whole situation.

47. General Principles. In order to establish the principles governing the position of the resultant force of gravity let us consider the simplest case as shown in Fig. 101: a timber supported by a rope around its center. Conceive of the timber as

* The resultant force of gravity is sometimes called the *resultant weight.*

composed of a number of elementary particles each acted upon by gravity and giving rise to a force dW, as shown. Now evidently the sum of the forces will be $\int_W dW$* and will be equal to the weight of the timber (call it W). The upward pull of the rope must equal this amount, since $\Sigma V = 0$. Moreover each elementary force dW sets up a moment of rotation about c equal to $z\,dW$. But since for each such positive moment there is an equal negative moment caused by a force dW symmetrically placed on the other side of c, it is evident that $\int z\,dW = 0$, when the origin for z is taken in the plane X.

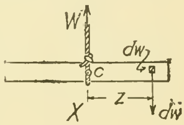


FIG. 101

Now let the timber in Fig. 101 be rotated in the plane of the paper about c, through an angle θ. The lever arm decreases, but the same decrease applies to the symmetrically placed particles on the opposite side of c, so that $\int z\,dW$ remains equal to zero. We can then say that the timber will remain at rest in any position so long as it is supported by a force equal to its weight W and which acts vertically in X and through c.

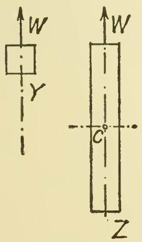


FIG. 102

If now we take the two views of the timber shown in Fig. 102, precisely the same considerations as before will lead to precisely the same conclusions with regard to planes Y and Z. The three planes X, Y, and Z intersect at the geometric center c of the timber, and from the above we can say that if the timber is supported from the point c it will remain in equilibrium in any position in which it is once placed because $\int_W z\,dW = 0$ in every such case. Moreover since the timber can be supported from c without rotation, it follows that the resultant force of gravity acts through c.

* The symbol $\int_W B\,dW$ will be employed to indicate that the integration is to be carried out for the entire weight W of the problem under discussion. In this symbol, B and dW both will ordinarily be expressed in terms of another variable of integration at a later stage of the solution. When this is done, the result will appear in the more familiar form of the integral between limits, as for instance, $\int_0^r r^2x\,dx$, page 72. A similar symbol will be employed in the cases of areas, volumes etc.

In the case of an unsymmetric body it can also be shown that there is a point through which the resultant force of gravity acts.* If this point is used as a point of support, the body will remain at rest no matter in what position it is first placed. This point about which a body may be balanced in any position and which is the point of application of the resultant force of gravity is called the ***center of gravity*** of the body.

In the case of symmetric bodies, the center of gravity lies at the geometric center. In the case of unsymmetric bodies, its location may be determined as shown in the following sections. Sometimes the center of gravity is a point in space related to but not a part of the body itself, as in the case of the chair in Fig. 100, or in the case of a bowl. But in every case where there is one or more planes of symmetry, the center of gravity will lie in that plane or those planes.

48. Two Axes of Symmetry. A. METHOD OF ADDITION. Since the body shown in Figs. 103 and 104 is symmetric about two axes, we can say by inspection that its center of gravity lies in the line

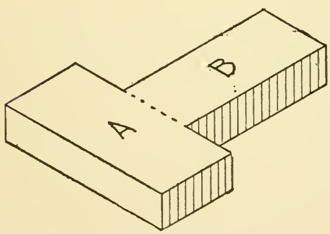


FIG. 103

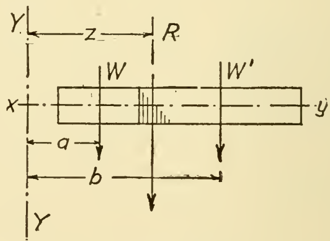


FIG. 104

xy. Moreover since each of its parts A and B is symmetric, the resultant weights of these parts can be located by inspection. Let these weights be W and W', as shown in Fig. 104, and let Y be any plane of reference, chosen arbitrarily. Now the forces W and W' constitute a set of parallel forces and their resultant can be determined as in §§ 41–43. Thus if R is the resultant

* Proof of this statement may be found in any more extended treatise on mechanics.

of W and W', and z is the unknown distance of R from Y, we have

$$(1) \qquad R = W + W',$$

$$(2) \qquad Rz = Wa + W'b.$$

Now substituting the value of R as given by equation (1) in equation (2), we find

$$(W + W')z = Wa + W'b,$$

whence

$$z = \frac{Wa + W'b}{W + W'},$$

which gives the location of the center of gravity in terms of known amounts. Compare with §§ 43–44.

B. Method of Subtraction. Let the body shown in Fig. 105 be conceived as a parallelopiped $10'' \times 8'' \times 1''$, minus two holes each $6'' \times 3'' \times 1''$. Let the body weigh $\frac{1}{2}$ lb. per cu. in., and let it be required to locate the center of gravity with reference to the left-hand face.

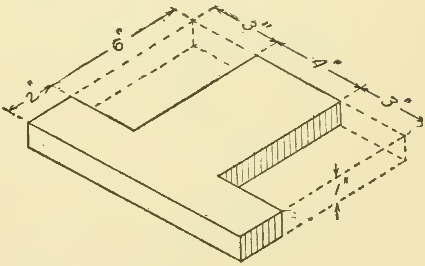


Fig. 105

The weight of the entire solid (including dotted parts) is $(1 \times 10 \times 8 \times \frac{1}{2}) = 40$ lbs., which acts downward at 4″ from the left face. The weight of the material cut out by the holes will be

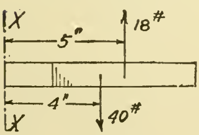


Fig. 106

$$-\,2[6 \times 3 \times 1 \times (\tfrac{1}{2})] = -\,18 \text{ lbs.},$$

acting at 5″ from the left face. Let these forces be shown in elevation as in Fig. 106, X being taken as a plane of reference. Since the effect of the holes is negative with respect to the solid part, the force of 18 lbs. is indicated as acting upward. The moment produced by these weights about

X can be determined as $(40 \times 4) - (18 \times 5) = 70$ lbs. ins. The resultant weight is evidently $40 - 18 = 22$ lbs., and it must be at such a distance from X that *its* moment will equal that just determined. Calling this distance $\bar{x}$,* we have

$$22\bar{x} = 70, \qquad \text{or} \qquad \bar{x} = 3.18''.$$

NOTE. At this point the student should assign a different set of dimensions to a solid similar to that in Fig. 103 and proceed to find its center of gravity, first by addition and then by subtraction. Of course the results should check.

PROBLEMS

NOTE. Weights of materials when needed may be determined from Table I in the appendix.

1. Figure 107 represents a block of stone weighing 130 lbs. per cu. ft. Find its center of gravity (*a*) by addition, (*b*) by subtraction.

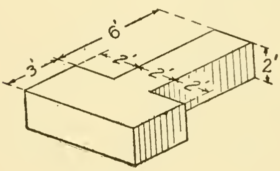


FIG. 107

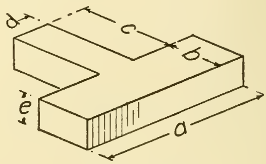


FIG. 108

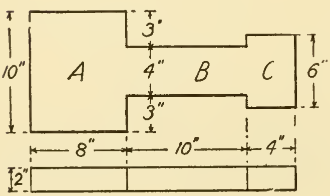


FIG. 109

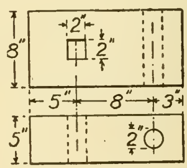


FIG. 110

2. In Fig. 108, (*a*) locate the center of gravity with reference to the right face, (*b*) with reference to the left face, (*c*) check results—material weighs $\frac{1}{4}$ lb. per cu. in.
3. Find the center of gravity of the body in Fig. 109, if the material of part A weighs 0.23 lb. per cu. in.; part B, 0.17 lb. per cu. in., and part C, 0.33 lb. per cu. in.
4. Find the center of gravity of the body in Fig. 110.

* Hereafter this symbol made up of a letter under a bar will often be used to designate distance to the center of gravity, and may be read "gravity x."

5. Find the center of gravity of a plate of wrought iron, 12″ × 12″ × $\frac{3}{4}$″, with a hole 2″ in diameter, the center of which is 3″ from the center of the plate and on one of its diagonals.
6. Let the hole in the plate in problem 5 be plugged with lead. Find the center of gravity.

49. One Axis of Symmetry. Let the angle bar shown in Fig. 111 be homogeneous and weigh $\frac{1}{4}$ lb. per cu. in. Let it be required to find the center of gravity. For that purpose, let the bar be divided into two parts as indicated by the shading on the end elevation. Since the bar is symmetric about the plane Z, the center of gravity is known to lie in this plane.

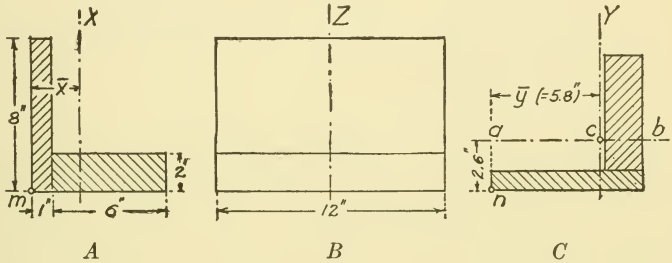


Fig. 111

Turning now to view A it will be possible to locate a plane X in which the center of gravity must lie by using either of methods of § 48. The weights involved are

Vertical leg:	8″ × 1″ × 12″ × $\frac{1}{4}$ lb. per cu. in. =	24 lbs.
Horizontal leg:	6″ × 2″ × 12″ × $\frac{1}{4}$ lb. per cu. in. =	36 lbs.
Entire angle bar		60 lbs.

Now taking a center of moments at m we find that these weights produce the following moments:

Vertical leg:	24 lbs. × $\frac{1}{2}$″ =	12 lbs. ins.
Horizontal leg:	36 lbs. × 4″ =	144 lbs. ins.
		156 lbs. ins.

This moment is the same as that of the entire weight (60 lbs.) acting in the plane X, whose distance $\bar{x}$ from m is to be determined. Equating the moments, we find

(1) $$156 = 60\bar{x}, \qquad \text{or} \qquad \bar{x} = 2.6''.$$

The center of gravity of the bar is now located as lying in plane X and in plane Z. It will therefore be in their line of intersection, which is shown in view C as the line ab. In order to determine which point on ab is the center of gravity, we will turn the angle bar as shown in view C and determine a plane Y which must contain the center of gravity, using the same method as above and computing moments about n, as follows:

Horizontal leg:	24 lbs. × 4″ =	96 lbs. ins.
Vertical leg:	36 lbs. × 7″ =	252 lbs. ins.
Entire angle		348 lbs. ins.

(2) $$60\bar{y} = 348, \qquad \text{or} \qquad \bar{y} = 5.8''.$$

This computation finally locates the point c as the center of gravity of the bar. The above computation could have been as well performed by method B, § 48. The student should note that the length of the prism (12″) and the heaviness of the material ($\frac{1}{4}$ lb. per cu. in.) enter as factors on both sides of equations (1) and (2) above. In solving the equations these factors cancel out. The meaning of this evidently is that the position of the center of gravity as determined by planes X and Y does not depend on the length of the prism nor on the heaviness of the material. It would therefore have been possible to get our results if those terms had been omitted from the computations. We would then have been dealing with areas instead of weights; but since these areas are *proportional* to the weights involved, the results would have remained the same. It will be good practice for the student now to assume a figure similar to Fig. 111, but with different dimensions, and find its center of gravity without using lengths or heaviness, and to make the computations by the subtraction method.

50. No Axis of Symmetry. Let it be required to find the center of gravity of the non-homogeneous block shown in Fig. 112. The weights of various parts of the block are as follows:

A	$= 10 \times 6 \times 2 \times \ 80 =$	9600 lbs.
B	$= \ 3 \times 6 \times 1 \times 100 =$	1800 lbs.
C	$= \ 3 \times 4 \times 5 \times 120 =$	7200 lbs.
Total		18,600 lbs.

Three equations of moment are required to locate the center of gravity, each corresponding to one of the positions of the block as shown in the elevation drawings:

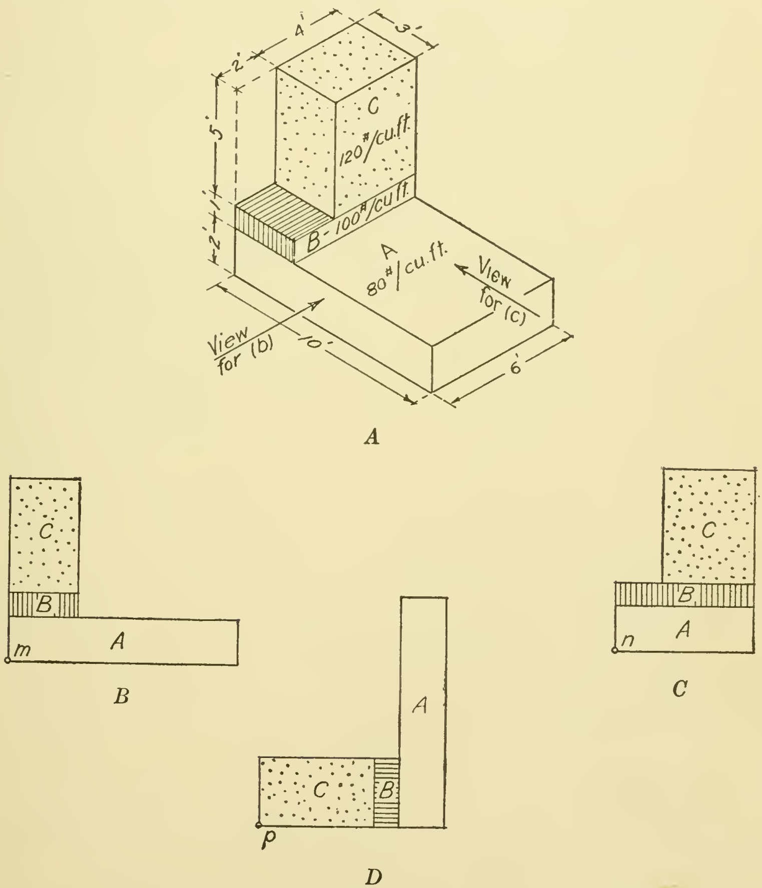


Fig. 112

view B; center of moments at m

$$(9600 \times 5) + (1800 \times 1\tfrac{1}{2}) + (7200 \times 1\tfrac{1}{2}) = 18{,}600\bar{x},$$

$$\bar{x} = 3.3',$$

view C; center of moments at n

$$(9600 \times 3) + (1800 \times 3) + (7200 \times 4) = 18{,}600\bar{y},$$

$$\bar{y} = 3.4',$$

view D; center of moments at p

$$(7200 \times 2\tfrac{1}{2}) + (1800 \times 5\tfrac{1}{2}) + (9600 \times 7) = 18{,}600\bar{z},$$

$$\bar{z} = 5.1'.$$

PROBLEMS

1. Find the center of gravity of the flitched timber in Fig. 113.
2. In Fig. 113, let the channels B and C be removed. Find the center of gravity. See a handbook for the details of the steel channel sections.

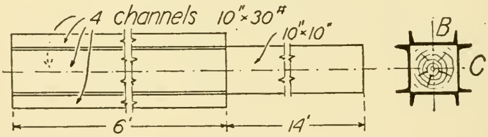


FIG. 113

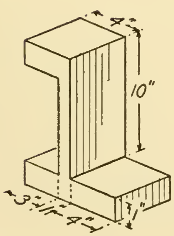


FIG. 114

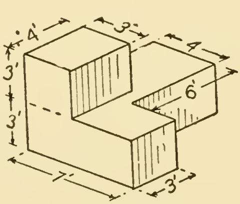


FIG. 115

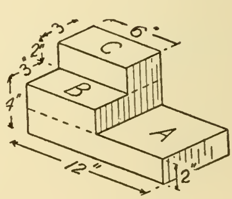


FIG. 116

3. Find the center of gravity of the solid in Fig. 114.
4. Find the center of gravity of the solid in Fig. 115.
5. Find the center of gravity of the solid in Fig. 116. Let A weigh $\frac{1}{4}$ lb. per cu. in., let B weigh $\frac{1}{2}$ lb. per cu. in., and let C weigh $\frac{1}{8}$ lb. per cu. in.

51. Various Solids. In each of the cases studied it has been possible to divide the body into parts, each of which is wholly symmetric, and the methods used have been made to depend on this fact. Many cases arise where this is not possible. Some solutions for such cases are indicated below.

A. BY INTEGRATION. In many regular homogeneous solids, particularly those determined by curved surfaces, the center of gravity can be determined readily by integration. The general type of solution in such cases is as follows:

Let V be the volume of the solid and let dV be any elementary part of that volume at the distance x from an arbitrarily chosen plane of reference. Also let $\bar{x}$ be the distance of the center of gravity of the volume from the same plane of reference. Then, from §§ 43 and 47, $V\bar{x} = \int_V x\,dV$. The solution normally hinges on selecting an elementary slice in such a way as to make it readily possible to express its volume in terms of x.

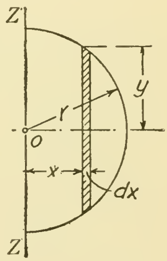


FIG. 117

As an example of the above let it be required to find the distance of the center of gravity of a hemisphere from its flat face, Fig. 117. Let the origin be at o. Let the volume be V and let dV be chosen as a thin section parallel to the plane Z–Z, r being the radius of the sphere. Now $V = \frac{2}{3}\pi r^3$; dV = volume of elementary slice $= \pi y^2\,dx$. Substituting these values in the general expression of the previous paragraph, we have

$$(1) \qquad \tfrac{2}{3}\pi r^3\bar{x} = \int \pi x y^2\,dx.$$

We also have the relation

$$x^2 + y^2 = r^2, \qquad \text{or} \qquad y^2 = r^2 - x^2.$$

Substituting this in (1), we get

$$\tfrac{2}{3}\pi r^3\bar{x} = \int \pi x\,dx(r^2 - x^2)$$

and, since x may vary between r and o, we have

$$\tfrac{2}{3}r^3\bar{x} = \int_0^r r^2x\,dx - \int_0^r x^3\,dx,$$

$$\tfrac{2}{3}r^3\bar{x} = \frac{r^4}{2} - \frac{r^4}{4},$$

$$\bar{x} = \tfrac{3}{8}r.$$

This single example will suffice to show the general method of this type of solution. Many other cases are to be found worked out in more extended treatises, and the results of such computations are to be found in many handbooks.

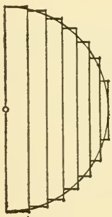


FIG. 118

B. BY APPROXIMATION. Sometimes a curved surface may be approximated by surfaces of simpler form. For instance the hemisphere of Fig. 117 might be approximated by parallelopipeds or cylinders, as shown in Fig. 118. The center of gravity could then be found by § 48. The degree of accuracy in the result would depend upon the closeness with which the true shape is approximated by the shapes assumed in the calculations.

C. BY TRIAL. The center of gravity of a very irregular body is often most readily determined by trial. The body may be suspended in several positions and vertical planes passed through the point of support for each position. These planes will intersect, and the point of intersection is the center of gravity of the body. The degree of accuracy attainable in this manner depends largely upon the facilities available and the care with which measurements are made.

PROBLEMS

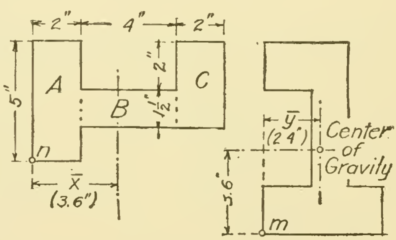
1. Find, by integration, the center of gravity of a square pyramid.
2. Find, by integration, the center of gravity of a cone whose base is 6″ in diameter and whose altitude is 12″.
3. Outline a method for finding the center of gravity of a frustum of a cone or pyramid.

52. Areas. At first it seems absurd to speak of the center of gravity of an area since an area has no weight and cannot be

affected by gravity. However, if we consider the center of gravity of an homogeneous prism, as presented in § 49 (particularly the concluding paragraphs), it will be evident that the position of the center of gravity, *as shown on view* C, Fig. 111, is in no way influenced by any factor except the size and the shape of the area. Moreover, if the prism is considered to vary in length, approaching an area as its limit, the center of gravity of the prism approaches the point c as its limit. This point c is called the center of gravity of the area. It is as much a property of the area as is the length, periphery, or any other property.

In § 47 it was shown that in the case of any solid, $\int z\,dW = 0$ when the origin for z is taken through the center of gravity of the solid. It will be evident that in the case of an area the same principle holds good and that $\int_A z\,dA = 0$ when the origin for z is taken through the center of gravity of the area.

The method of determining the center of gravity of an area is identical in principle with that of § 49. The computations are the same except the terms length and heaviness are omitted, the areas being used as if they were weights. A few typical solutions are given below.



A B

Fig. 119

A. By Addition. The area shown in Fig. 119 is divided into parts, marked A, B, and C for convenience. Taking the center of moments at n, we find

		Area		Lever Arm		Moment
(A)	$5 \times 2 =$	10	$\times$	1	=	10
(B)	$4 \times 1\frac{1}{2} =$	6	$\times$	4	=	24
(C)	$2 \times 3\frac{1}{2} =$	7	$\times$	7	=	49
	Total	23	$\bar{x}$		=	83
			$\bar{x}$		=	3.6″

Using the view B with the center of moments at m, we find

	Area		Lever Arm		Moment
(*A*)	5×2	$=$	$10 \times 2\frac{1}{2}$	$=$	$25,$
(*B*)	$4 \times 1\frac{1}{2}$	$=$	$6 \times 2\frac{3}{4}$	$=$	$16.5,$
(*C*)	$2 \times 3\frac{1}{2}$	$=$	$7 \times 1\frac{3}{4}$	$=$	$12.25,$

$$23\bar{y} = 53.75,$$

$$\bar{y} = 2.337 = 2.4''.$$

B. By Subtraction. Figure 120 gives the following:

Area		Lever Arm		Moment
4×8	$=$	32×4	$=$	$128,$
$-\pi(1.5)^2$	$=$	-7.07×2	$=$	$-14.14,$

$$24.93\bar{x} = 113.86,$$

$$\bar{x} = 4.56''.$$

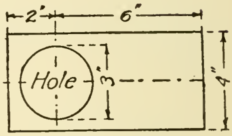


Fig. 120

C. By Integration. It is not always possible to divide an area into parts, whose centers of gravity can be determined by inspection, as was done in A and B, above. In such case, if the outlines of the area are regular, so as to establish mathematical relations, the center of gravity may be determined by integration.

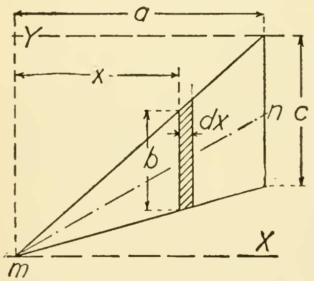


Fig. 121

The principle of all such solutions is the same, though the detail of the operation will vary considerably with the area in question. For the purpose of illustration, let us take the simple case of the triangle shown in Fig. 121.

If the figure has a line of symmetry, the center of gravity will lie on that line. In the case in hand the median line, *mn*, is such a line. After the line of symmetry (if any) is noted, the figure is placed in some convenient relation to a pair of coordinate axes. In this case, the vertex is touching and the base is parallel to the y axis. Next the figure is divided into elementary strips of area of convenient shape. In this case strips parallel to the base are chosen (shown shaded in the figure). Now let dA be

the area of any such strip distant x from the y axis. The moment of this strip about the y axis will then be $x\,dA$, and the total moment for the entire figure will be $\int_A x\,dA$. Now if A is the area of the triangle and $\bar{x}$ is the distance of the center of gravity from the y axis, the principles of §§ 44–47 give the equation

$$A\bar{x} = \int_A x\,dA. \tag{1}$$

This is the general form from which all such computations for center of gravity start. It is now necessary to establish a value for dA which can be expressed in terms of x.

From the figure we have

$$dA = b\,dx, \tag{2}$$

and from similar triangles

$$\frac{b}{c} = \frac{x}{a}, \qquad \text{or} \qquad b = \frac{c}{a}x.$$

Substituting this in (2), we find

$$dA = \frac{c}{a}x\,dx;$$

and substituting now in (1), we have

$$A\bar{x} = \int_A \frac{c}{a}x^2\,dx.$$

Now x may have all values between o and a. Inserting these limits and expressing A in terms of c and a, we have

$$\tfrac{1}{2}ca\bar{x} = \int_o^a \frac{c}{a}x^2dx.$$

Now evaluating the integral, we obtain

$$\tfrac{1}{2}ca\bar{x} = \tfrac{1}{3}ca^2, \qquad \text{or} \qquad \bar{x} = \tfrac{2}{3}a.$$

D. By Approximation. The area in Fig. 122 could be approximated by a number of rectangles as shown and the center of gravity located by computations as in A, above.

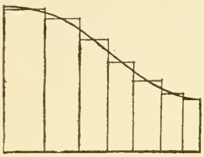


Fig. 122

E. By Trial. Cut from a thin stiff sheet an area of the required shape. Let it hang freely from any selected point and, while so suspended, draw a vertical line through the point of support. Choose a second point of support and draw a second vertical line through it. The point where the two lines so determined intersect is the center of gravity. It will usually be wise to check this point by drawing a third line in the same manner; this should check with those previously drawn.

The center of gravity of a triangle can be located by purely geometric means. A line from a vertex to the center of the opposite side (i.e., a median) is a line of symmetry. All three such lines meet at a point, which is the center of gravity and which lies at the third point of each median line (as proved above). See Fig. 123.

53. Static Moment. The quantity $\int_A x\, dA$, used in equation (1), § 52, is known as the static moment of the area. It represents the product of an area multiplied by its distance from a chosen axis. When the area used is a finite one, the distance of the area from the axis (x) is measured from the center of gravity of the area. Thus the static moment of an area may be said to be the area times its *average* distance from an axis. The axis may be chosen arbitrarily.

In § 51, the expression $\int_V x\, dV$ is the static moment of a volume. Its significance can be easily apprehended from the explanation given in the preceding paragraph. In the case of a volume the distance x is measured from a plane of reference instead of a line.

The term static moment implies a force acting on a lever arm and hence it is not logically applicable to an area any more than is center of gravity (§ 52). However, it is commonly used in the manner explained above.

54. Summary. From the preceding paragraphs (§§ 44–53), the student should have become familiar with the principles governing the determination of center of gravity. However, it may be well to point out that every solution is essentially an integrating process dependent on the principle of § 44. When the case in hand involves a body or an area that can be divided easily into symmetric components, the computation is simplified, as has been illustrated frequently; but in these cases as well as in the others, the underlying principle is to be found in the equation

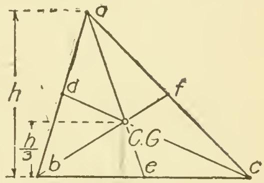


FIG. 123. Center of gravity of a triangle.
$ad = db$
$be = ec$
$cf = fa$

$$V\bar{x} = \int_V x\, dV,$$

as developed in § 51.

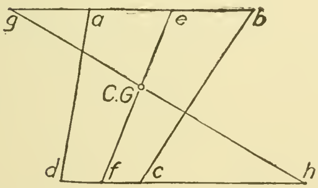


FIG. 124. Center of gravity of a trapezoid
$ae = cb$
$df = fc$
$ag = dc$
$ch = ab$

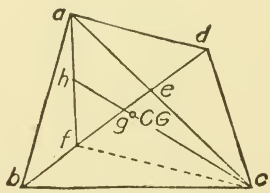


FIG. 125. Center of gravity of a quadrilateral
$bf = ed$
$ah = hf$
$hg = \frac{1}{3}hc$

Centers of gravity of many solids and areas have been worked out and the results of such computations are available in books of reference. Figures 123–125 illustrate graphic solutions for centers of gravity that will be found useful.

PROBLEMS

1. Find the center of gravity of the area, Fig. 126.
2. Derive the values of x and y, Fig. 127.

3. Find the center of gravity of the area, Fig. 128.
4. Find the center of gravity of the area, Fig. 129.
5. Find the center of gravity of the area, Fig. 130. (Use handbook.)

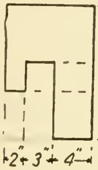


Fig. 126

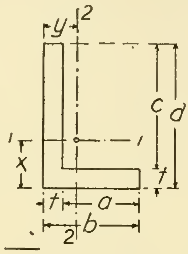


Fig. 127

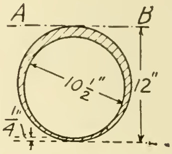


Fig. 128

6. Test the result of problem 4 experimentally. Work at full size.
7. Derive a formula for the center of gravity of (*a*) a trapezoid; let the altitude be h and let the greater base be B while the lesser base is b; (*b*) a hollow half circle; let the outer diameter be D and the inner diameter d.
8. Derive, by integration, the center of gravity of a semi-circle.

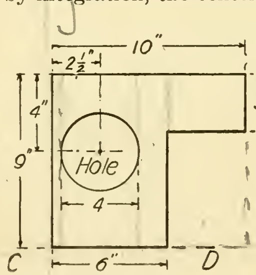


Fig. 129

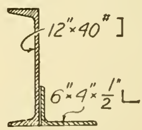


Fig. 130

9. Work out a proof of the rule for finding the center of gravity of a trapezoid, as given in Fig. 124, using the results of problem 7 (*a*).
10. Work out a geometric proof of the rule for finding the center of gravity of a quadrilateral, as given in Fig. 125.

CHAPTER VII

UNIT STRESSES

55. General. If a cable such as is shown in Fig. 131, composed of 100 separate wires, is made to sustain a load of 10,000 lbs., each wire will have to withstand on the average a stress of 100 lbs. If there were more wires, the individual stress would be less. All materials of construction may be considered to be made up, somewhat like a cable, of small individual fibers each carrying its proportion of the total stress on the piece of which it is a part.* Where there are many of these fibers (i.e., when the area of the cross section is large) the stress on each is small, and vice versa. The average stress per fiber evidently will be found by dividing the total stress by the number of fibers. For the purpose of standardization and ease of computation we usually work with these elementary fibers in a wholesale way, that is, as many as are needed to make up one square inch of cross section are dealt with at once, and the resultant stress per fiber (fiber stress) is expressed not in terms of pounds per fiber but as pounds per square inch of cross section. Thus if a bar 2″ × 2″ carries a load of 10,000 lbs., the fiber stress is 10,000 lbs. divided by 4 sq. in. = 2500 lbs. per sq. in. This result is sometimes called the *fiber stress*, or the *unit stress* (i.e., stress per unit of area of cross section), or the *intensity of stress.* It is most frequently expressed in lbs./sq. in., but other units such as lbs./sq. ft. and tons/sq. ft. are sometimes used.

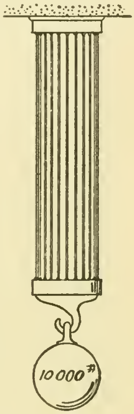


Fig. 131

56. Axial Loads. The materials of construction usually occur in forms that have distinct axes of symmetry. The longest axial

* The student should not get the impression that the materials of construction are in general actually fibrous in their make-up. The fiber idea is merely a convenient concept.

line is designated as the *longitudinal axis* (or sometimes merely as "the axis"). A section taken perpendicular to such an axis is called the *cross section.* When a load is placed on any structural member in such a way that its line of action coincides with an axis of the member, as in Fig. 23, the load is called an *axial load,* and, if the material is homogeneous, the resulting stresses are uniformly distributed over the cross section, as shown in Fig. 132. The amount of such unit stress is determined by simple division, as illustrated in the previous article.

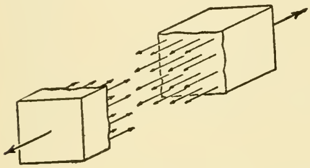


FIG. 132

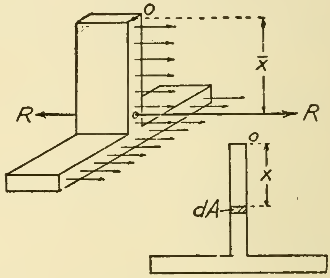


FIG. 133

If the cross section of any structural member is unsymmetric, a uniform stress distribution will result from a load acting along the member in a line that cuts each cross section at its center of gravity.*

* If proof of this statement is required, let the case shown in Fig. 133 be considered. Let the unit stress be the same over the entire section and let the position of the resultant load be required. Let A be the area of the section, R the amount of the resultant load, $\bar{x}$ the distance of R from an arbitrary center of moments o, dA an elementary area distant x from o and let S be the unit stress on dA (represented by one of the arrows in Fig. 133). Then

$S\,dA$ = The total stress on an elementary area.

$\int S\,dA$ = The total stress on the section.

$\int Sx\,dA$ = The moment of the stress on the section, about o.

From $\Sigma M = 0$;

$$R\bar{x} = \int_A Sx\,dA = S\int_A x\,dA \text{ (since } S \text{ is a constant);}$$

$$\frac{R}{S}\bar{x} = \int_A x\,dA.$$

But $R/S = A$ (by definition); then

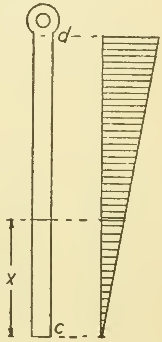
$$A\bar{x} = \int_A x\,dA.$$

We may then conclude:

(1) *Any load which is applied along the center of gravity of the cross section of a homogeneous bar will cause uniformly distributed stresses throughout the bar, and vice versa.*

(2) *The uniform unit stress produced by such a load is found by dividing the load by the area of the cross section.*

57. Stresses due to Own Weight. If a bar such as that shown in Fig. 134*A* is suspended from a pin, it will be subject to tensile stresses due to its own weight. The unit stresses involved evidently will be zero at the bottom and a maximum (equal to the weight of the bar divided by its cross-sectional area) at the top. Evidently these stresses will increase uniformly from zero at c to a maximum at d. This variation is represented graphically in Fig. 134*B*.



A B

FIG. 134

If the area of the bar is a sq. in., and its heaviness (weight per unit volume) is b lbs./cu. in., the stress at any point distant x from the lower end will be abx lbs., and the unit stress at the same point will be bx lbs./sq. in.

If such a bar carries a load of P lbs. at its lower end, the unit stress at the section x will be

$$(P/a + bx) \text{ lbs./sq. in.}$$

Actual cases where the weight of a member causes an important tensile stress in the member itself will be few. A very long elevator cable might be a case in point.

In compression, the stresses in high walls or piers are frequently due in large part to their own weight.

58. Pier Uniformly Stressed. From the preceding article we may infer that in a pier of uniform section the bottom is under a greater unit stress than the top owing to the weight of material from which the pier is built.

From § 52 we know that this is the condition which is fulfilled when $\bar{x}$ represents the distance from the center of moments to the center of gravity of the section. Therefore, if a load acting on a bar is to produce uniformly distributed unit stress on the cross section, it must act at the center of gravity of the section.

It is evident that if the section at the bottom is properly increased the unit stresses at top and bottom may be made equal. Moreover, if just the proper form is used, the unit stress at all heights may be kept the same. It is a matter of some interest to determine the theoretical shape of a pier that is uniformly stressed, for while one would seldom or never be justified in building such a shape, it may be more or less closely approximated. Again the study of this problem will lead up to others of more immediate importance.

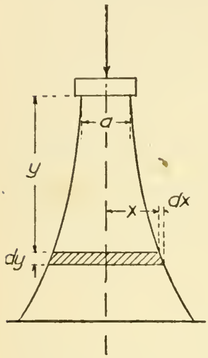


FIG. 135

Let the pier be everywhere square in section and let its elevation be shown by Fig. 135.* We can now proceed to determine the nature of the curve bounding the sides. Let the central load be P and the area of the top be a^2. Let the heaviness of the material in the pier be h and let the uniform unit stress be s. This unit stress $s = P/a^2$; and it will be the same on every section of the pier. Choose any elementary section of the pier, as shown shaded in the figure. Now the unit stress at the top of this section is P/a^2 and at the bottom of the section it must be the same. Therefore the difference in the areas at top and bottom must be just sufficient to support the weight of the slice between at the specified unit stress. Putting this in actual figures, we find

$$\text{Area of top of section} = (2x)^2 = 4x^2.$$

$$\text{Increase in this area at bottom} = d(4x^2) = 8x\,dx.$$

$$\text{Weight of section of pier} = (2x)^2 h\,dy = 4x^2 h\,dy.$$

Therefore

$$\frac{4x^2 h\,dy}{P/a^2} = 8x\,dx, \qquad \text{or} \qquad dy = \frac{2P}{a^2 hx}\,dx.$$

* At this point the student should assure himself that the outline will actually be a curve of some sort and prove it to himself before proceeding.

Integrating this and determining the constant from the fact that $x = a/2$ when $y = 0$, we get

$$y = \frac{2P}{a^2h} \log_e \frac{2x}{a} \cdot$$

This equation shows the character of the curved outline. Figure 136 is drawn approximately to scale to show a case where actual values have been chosen for P, a, and h, and the computations worked through.

59. Net Sections. As a chain is no stronger than its weakest link, so a structural member is no stronger than its smallest section. Thus the strength of a threaded bolt is not based on the full sectional area, but rather upon the area at the root of

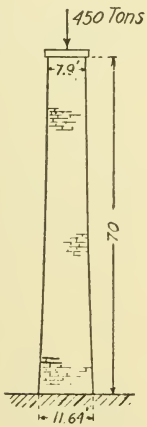


FIG. 136

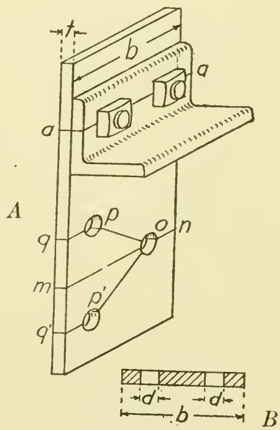


FIG. 137

the thread. (See table of screw threads in any handbook.) Again if holes are cut in a member for any purpose, the reduced section limits the strength of the whole piece. In Fig. 137A an angle shelf is bolted to a steel plate. The holes punched out for the bolts reduce the section which before punching was equal to bt. The reduced section cut through the holes is shown in Fig. 137B. Its area is $t(b - 2d)$.

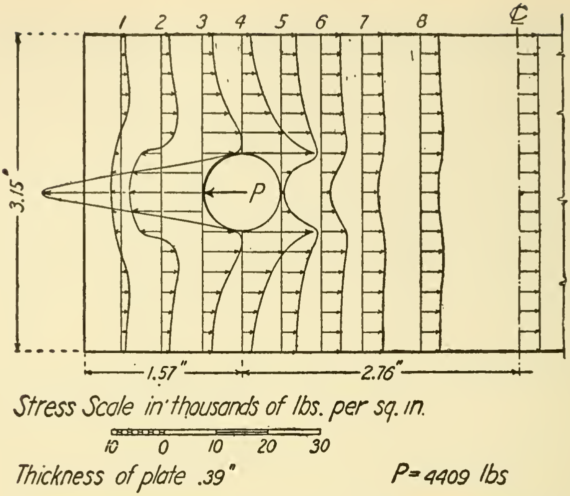


FIG. 138A. Graphical illustration of test results on a plate, half of which is shown. Tension was applied by means of bars through the holes at either end. Note compression behind hole and tension elsewhere in varying amount. (D. Rühl; ZEITSCHRIFT V.D.I., vol. 64, p. 549).

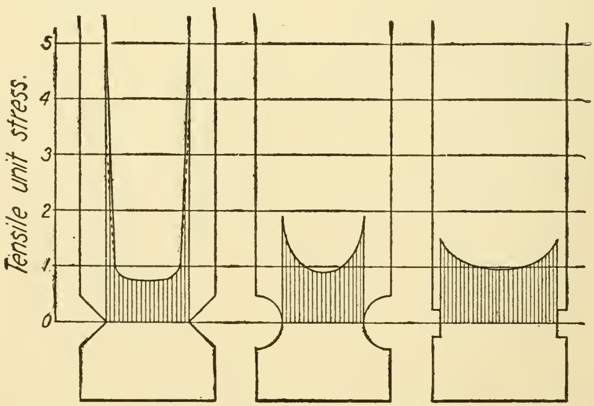


FIG. 138B. Relative amounts of stress across the minimum sections of rectangular bars under tension. (E. Preuss; ZEITSCHRIFT V.D.I., vol. 57, p. 664).

If holes are "staggered" (as shown at section *nopq*, in Fig. 137*A*), manifestly the plate will be stronger than when the holes are opposite, as at section *aa*. The maximum strength of the

punched plate would be given by such a section as *mn*. The distance *mq* should, if possible, be such as will make the section *nopq* (which passes through two holes) at least equal to *mn* (which passes through a single hole). See the footnote on page 131.

When the area of a bar is suddenly changed at a given section, as in cutting a thread on a rod or in punching a hole in a plate, the unit stresses at that section are not uniformly distributed even though they may be uniform on the sections above and below the one which is reduced. The stresses tend to become concentrated at the edges of the cut, as shown in Fig. 138*B*. Ordinarily this concentration of stress at a reduced section is considered as being cared for by the factor of safety, but in important computations or when precise results are demanded, it should be allowed for. This matter is treated quite fully in some of the more extended texts.

60. Unit Strength. It is a matter of common experience that a small steel wire is stronger than a much thicker rope, or if of equal thickness the wire will carry the greater load, i.e., the material of which it is made has a greater ***unit strength.*** In order to design structural parts economically and safely, it is necessary that we know the relative and also the absolute strength of the various materials that may be used. In order to determine these amounts many tests have been made, so that plenty of data is available. The unit stress *which will just break* a given material is called its ***ultimate strength.***

The results of such experiments are not by any means precise. A considerable variation is found between the ultimate strengths of pieces of material that are apparently alike. Thus any tables, such as those in the handbooks or in the appendix of this book, should be regarded as giving average values rather than precise ones.

The strength of a given material may vary widely under different kinds of stress. This fact is especially well illustrated in such materials as wood, concrete, and cast iron. Again if the material has a definite grain, as in wood, its strength may depend on the direction of the applied forces, relatively to the grain of

the material. These questions are more fully discussed in Chapter IX.

61. Factor of Safety. In designing a structure no piece of material is ever proportioned to carry a stress equal to its full ultimate strength. For instance, a bar of steel 1″ square may have an ultimate strength of 64,000 lbs., in tension, but, ordinarily, it would not be designed to carry more than 16,000 lbs. In such a case the factor (4) which exists between the load (or unit stress) for which the structure is designed and the ultimate possible load (or unit stress) is called the ***factor of safety.***

The use of a factor of safety is necessitated by a number of considerations from which the element of uncertainty cannot be eliminated. A structure can be designed to carry a definite load, but the designer cannot guard against overloading due to carelessness or ignorance. Individual pieces of material vary more or less in structure and hence in strength. Average strength values are used in design, but the weakest piece must be able to carry its load. Much of the data in regard to strength of materials is based on tests of small-sized specimens. Large pieces do not always develop a proportionate strength.

Some materials are much more uniform in quality and hence more dependable than others. Such materials can be used with relatively small factors of safety.* Another matter that has much to do with the use of a factor of safety is the phenomenon known as the fatigue of materials which is treated in § 85.

All or any of the above considerations are apt to enter into the determination of the factor of safety in any given case and sometimes the question of the permanency of the structure also enters. In a temporary building smaller factors may be used. Where unusual conditions favoring deterioration are present and a

* The following is quoted from Circular 295 of the U. S. Department of Agriculture, dealing with stresses on timber:

"The belief that a timber with a so-called 'factor of safety' of 3 or 4 will carry three or four times the load for which it is designed is erroneous and has caused many failures through overloading of structures. The application of a load which would produce three times the working stresses given in this circular would be expected to cause the immediate failure of some of the timbers, and the ultimate failure of 75 per cent of them. The application of loads which would produce stresses only one and one-half times the working stresses would be expected to cause occasional failures if the loads were left on for any great length of time."

permanent structure is desired, factors are larger. Quiescent loads admit of the lowest factors of safety. If the loads are such as to produce vibration or shocks, the factors of safety are increased or other means taken to insure safety.

The determination of factors of safety is a matter requiring mature judgment and a thorough understanding of the materials and loads to be used and the results which are to be attained. The smallest factor commonly used is 2, while factors of 3, 4, 5, and 6 are common; sometimes factors as high as ten to twenty are deemed necessary.

Table II in the Appendix gives some factors of safety that may be used in discussing the problems in this text. Especial attention is called to the note accompanying the Table and to the footnote on page 95.

The use of formal factors of safety is generally being superceded by the adoption of working unit stresses, as explained in § 62 and as set forth in Table I in the Appendix.

62. Working Unit Stress. Where many calculations are made for a given class of work and for the same material, the factor of safety is, of course, the same for all. In such cases it is usual in the beginning to divide the ultimate strength by the appropriate factor of safety and thus derive a quantity called the ***working stress,*** which is then used throughout all the calculations. In such a case the ultimate strength and the factor of safety no longer appear, their place being taken by this derived quantity.

Illustration of ultimate and working unit stresses for different materials may be found in the various handbooks and in the tables in the Appendix.

Much structural designing is done in accordance with specifications or ordinances which fix the allowable or working unit stresses. Municipal building codes are not always the best possible guides for practice but the working unit stresses ordinarily have at least the advantage of being conservative.

PROBLEMS

NOTE. For strengths of various materials see Table I in the Appendix.

1. In Fig. 26A if the supports are of steel wire whose diameter is 0.134″, what is the unit stress on each? What is the factor of safety?
2. What size of bearing plates are required under the ends of the beam shown in Fig. 139.
3. In Fig. 38, the back-stay is made of a $\frac{3}{4}''$ diameter wrought iron rod (threaded). Is it safe under the load shown? Refer to a handbook for areas of threaded rods.
4. What is the unit compressive stress in each of the two end blocks shown in Fig. 140?
5. See Fig. 141. What are the necessary dimensions of the surface of contact between the various parts as determined by compressive strength—earth being able to carry 4 tons per sq. ft.?

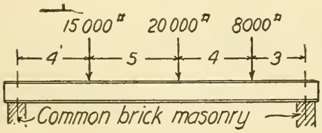


FIG. 139

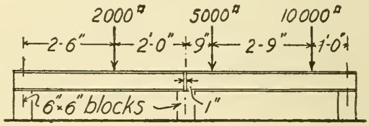


FIG. 140

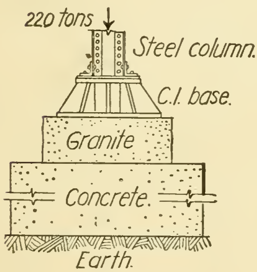


FIG. 141

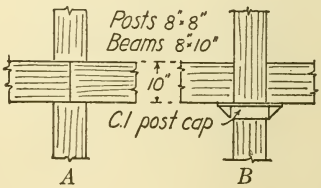


FIG. 142

6. What is the greatest safe load on the upper post in Fig. 142A and in Fig. 142B. The posts and beams are of wood.
7. Let the block shown in Fig. 110 be of timber with the grain running lengthwise of the block. (a) Let compressive forces of 25,000 lbs. be applied to the 5″ × 8″ ends. What is the factor of safety? (b) What is the safe load in compression if the forces are applied to the 8″ × 16″ faces of the block?
8. How high may a pier of brick masonry be built before it will crush under its own weight?
9. Draw a curve to show the variation in unit stress in a right circular cone resting on its base.
10. Draw a curve to show the variation in unit stress in a right circular cone hanging from its apex.
11. A pier of stone masonry in the form of a truncated pyramid is 8′ × 8′ square at the top and 75′ high. If it is to carry its full safe load on the top face, how large must it be at the bottom?

Unit stress = load / area cross section

Factor of safety = ult unit stress / work. stress

CHAPTER VIII

STRESS AND DEFORMATION

63. Introduction. The deformation of materials under stress is such a commonplace of daily life that its significance is often missed. The stretching of a rubber band, the springiness in the turf, the swaying of trees in a wind; these and a thousand other facts of daily life are accepted as the natural and everyday course of events. Yet most people are not so ready to accept the fact that all structures subjected to wind sway as do the trees, though to a lesser extent. They accept as natural the bending of a steel fishing rod, but they do not so readily conceive that a large column reacts in the same way.

As a matter of fact all materials, when subjected to stress, are more or less deformed. Painstaking investigation shows that, while the actual deformations may be small, such materials as steel and iron and even stone and concrete have quite definite elastic properties. They stretch under tension and shorten under compression. Increase in length is accompanied by a decrease in cross-sectional dimensions, while a shortening of length is accompanied by an increase of section. This lateral deformation is treated more fully in § 77. For the present the term deformation will be applied only to changes in length due to tensile or compressive stresses. When the stress is removed, the material returns (more or less) to its original shape. It is well to note that, technically, the quality of elasticity in a piece of material is judged not by *how much* it will deform, but by its *ability to recover* its original shape when stress is removed. However the actual amount of deformation is often of the greatest importance as there is a direct relation between it and the unit stresses which accompany it, as will be shown in what follows.

64. Testing. In gathering data as to the strength and elasticity of materials, many operations must be performed. Many

minute measurements must be made and many observations recorded in the course of a single test. Moreover, it is highly important that these measurements and observations be made in such a manner as to eliminate, in so far as possible, those errors and inaccuracies that always accompany human effort and which are known as the personal equation.

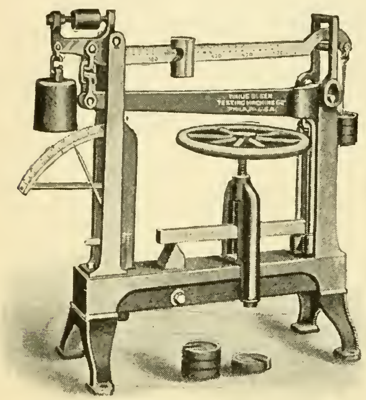


Fig. 143

The most obvious way of meeting the above difficulties is the employment of machinery, in so far as possible. These machines have been developed in great numbers and variety to meet widely differing conditions. Machines to produce tension, compression, shear, bending, torsion, etc. in amounts varying from a few pounds up to several million pounds have been developed. Also measuring devices, accurate to 1/10,000″, have been made in many different forms.*

A few of the many types of apparatus in general use are shown in Figs. 143 to 145. The elements common to nearly all such machines are well shown in Fig. 143 which is an Olsen hand power machine used primarily for testing cast iron bars in bending. Power is applied through the hand wheel. The right reaction is carried to a series of levers and finally weighed on the scales at the top. The quadrant at the left measures the deflection.

Figure 144 shows a Riehle 200,000 lbs. screw power machine for tension or compression tests. This machine is typical of a large variety of machines varying from 10,000 lbs. to 1,000,000 lbs. capacity. Power is supplied from any convenient source to a train of gears in the casing A, and these in turn operate the vertical screws B. The movable cross head, C, is actuated by the screws and may be made to travel either up or down. For

* With some types, deformations can be measured accurately to 0.00002 in.

tension tests, the specimen is inserted between the top head D and the cross head C. For compression tests the specimen is inserted between C and the weighing table E. In either event the load comes on the weighing table E, which is mounted on the

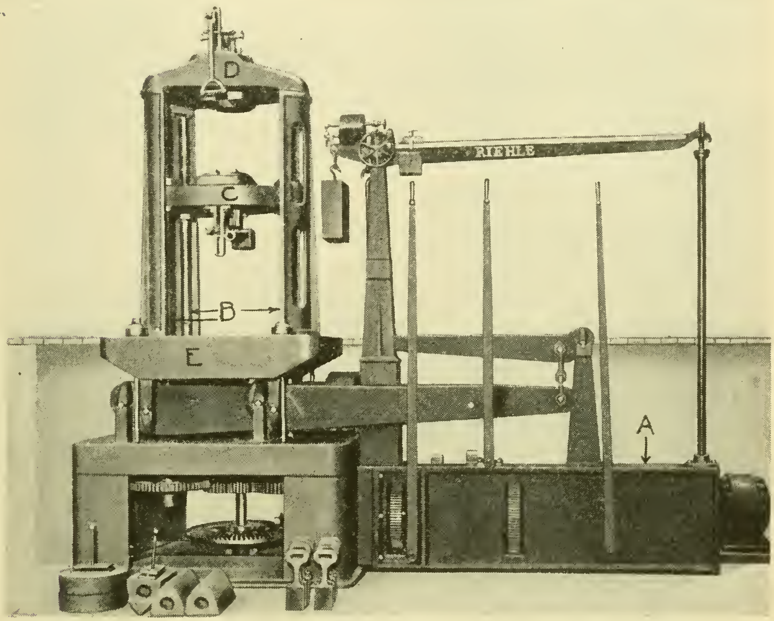


Fig. 144.

short ends of a train of levers which finally leads to the scale beam on which the load imposed on the specimen is weighed. The vertical levers on the gear casing are for shifting the gears so the power may be applied at any one of several rates, as desired. Various automatic and autographic devices may be added to keep the beam balanced during a test and to record the results. This type of machine can usually be adapted to cross bending and shear tests by the use of some simple additional parts.

Figure 145 illustrates the Olsen 10,000,000 lbs. machine which was built for the Bureau of Standards. This is the largest machine in use at the present time and was built primarily for

experimental work on full size specimens. The machinery above the movable head is used merely to adjust the head on the specimens. The power is applied hydraulically by machinery located below the floor and not shown in the illustration. The apparatus at the right controls both the application and the weighing of the load so that a single person can control both operations.

In making a series of tests with such apparatus countless details arise which demand careful attention. The size and shape of the test piece, its placing in the machine, the rate at which stress is applied; where and when measurements are to

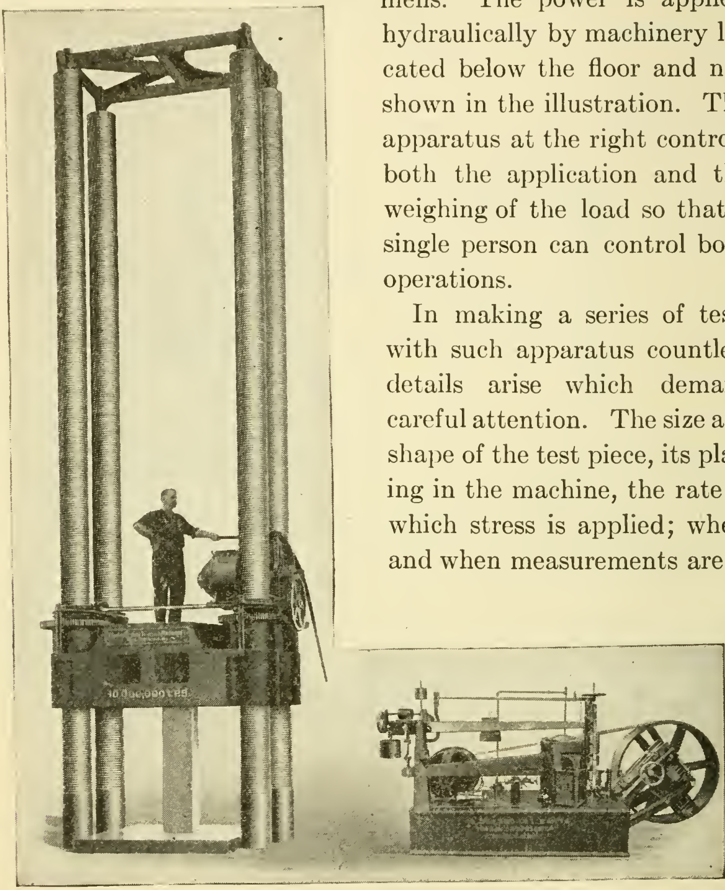


Fig. 145.

be taken; these and many other details have an important bearing on the ultimate meaning and value of the results obtained. Moreover it is important that the results of experimentation by different investigators be fairly comparable. Hence, they should have been arrived at by similar methods. These considerations have given rise to an elaborate technique in testing. The

American Society for Testing Materials is the outcome of an attempt to simplify and standardize this technique. Through its efforts and those of similar societies abroad, much has been accomplished. Today one can almost say that the testing of materials is a distinct science with a voluminous literature, a well-developed technique, thoroughly equipped and widely distributed laboratories and national and international correlating societies.

Obviously a designing engineer or architect can hardly hope to master the intricacies of so special a field. But he must know enough of its methods and its vocabulary to enable him to use its literature and its conclusions with ease and precision.

65. The Fundamental Relation. When a piece of elastic material is tested, the deformations which result from various

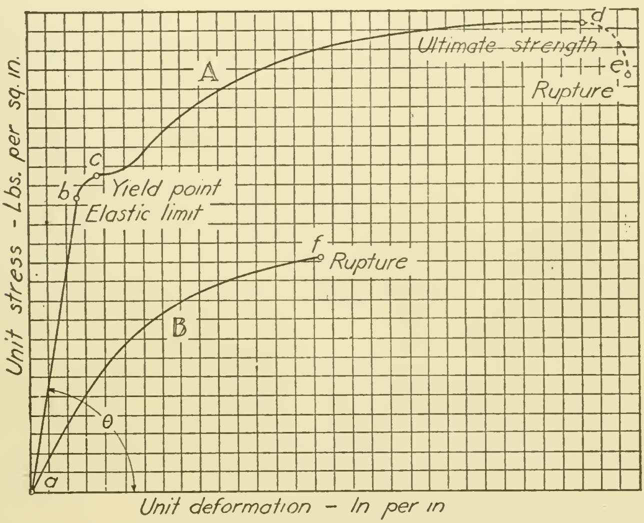


FIG. 146. Typical stress deformation curves. A for ductile materials. B for brittle materials.

unit stresses being carefully measured, a constant ratio is found to exist between the applied ***unit*** stress and the corresponding ***unit*** deformation (i.e., deformation per unit of length),* provided the unit stresses are within definite limits as explained in § 66.

* Unit deformation is found by dividing the total observed lengthening or shortening by the original length of the piece, both quantities being expressed in the

This ratio may be expressed by a curve. Figure 146*A* shows a curve drawn to represent *no definite material* but rather to illustrate the typical features of all such curves. The striking features are: the straight sloping line at the start, the short horizontal part, just following, and the uncertain character of the remainder.

It should be noticed that the same curve (but one drawn at different scales) would result from the use of *total* stresses and *total* elongations. Which method of representation is used in any given case is a matter of convenience only.

Table V in the Appendix gives the observations made during a typical test of steel and cast iron bars, in tension.

It will be found worth while to plot these results in the form of a curve similar to Fig. 146*A*.

66. Elastic Limit. It may be stated that, in general, all elastic materials which are under stress will return, upon removal of the loads, to their former length. This statement, however, is found to be true only when the unit stresses involved are less than a certain limit. Beyond this limit, the ***elasticity*** (i.e., the ability to resume its original shape) of the material is impaired. This is shown by the permanent deformation (set) which is found to result from these high stresses. This occurrence of permanent "***set***" marks, then, the limit of elasticity or ***elastic limit*** of the material.

At the same unit stress which marks the elastic limit (as determined by permanent set) there occurs also a change in the constant ratio between unit stress and unit deformation.* Deformations increase at a more rapid rate than applied loads. This fact is shown by the plotted curve (Fig. 146) leaving its constant direction and bending more rapidly toward the right at the point marked "elastic limit." The uncertain character of the curve beyond this point would seem to indicate, and investigations prove, that to all intents and purposes the elastic limit marks

same terms. Thus, if a bar of wrought iron 23′ 0″ long is stretched 0.138″ by a given load, the unit elongation is 0.138″ ÷ (23 × 12)″ = 0.0005. This quantity it will be noted is neither inches, feet nor pounds. It is a ratio; the ratio between original length and total deformation.

* More detailed texts show this statement to be only approximately correct.

incipient failure. This being the case, it will be apparent at once that the elastic limit is of great significance to the designer as marking the limit beyond which no part of any structure should ever be stressed.* The elastic limit might be expressed either in terms of unit deformation or the unit stress at which it occurs. The latter, however, is the accepted way, elastic limits being always stated in pounds per square inch.

67. The Yield Point. When ductile materials are tested at a stress slightly greater than the elastic limit, there comes a time when there is a sudden yielding or slipping as indicated by the horizontal portion of the curve in Fig. 146A. A relatively large deformation occurs without any increase in the load. This yielding is only momentary, however, and the material, having become adjusted, can again take up and hold an increased load.

In order to understand the significance of the yield point, the conditions of an actual test must be visualized. As the machine increases the load on the test piece, the operator keeps the scale beam in constant balance by running the poise on the scale beam outward. The increase of load and the movement of the poise are carefully synchronized to maintain the balance of the scale beam. When the sudden yielding of the specimen occurs, as above described, the scale beam drops and this automatically releases the load, or some part of it. During this period the test heads of the machine are moving and producing deformation but no additional stress. The unit stress at which this action occurs is called the ***yield point.*** Soon the scale beam rises again, indicating that the yielding is over and the operation continues as before.

In one sense the yield point is of no significance to the designer, since it lies above the elastic limit, while the stresses used in design are below the elastic limit. However, the yield point is relatively close to the elastic limit, and is much more easily determined in commercial testing. Therefore, there is a tendency to confuse the two, both in nomenclature and in the actual values quoted.

* Because of this it is usual to take the elastic limit into account when fixing factors of safety. (§ 78.)

The curve shown in Fig. 146*B*, is typical for non-ductile materials. The elastic limit is poorly defined and the yield point is non-existant.

68. The Ultimate Strength. Beyond the elastic limit, as before noted, the behavior of the material is altogether uncertain. No generalities can be stated as to the amounts of deformation or

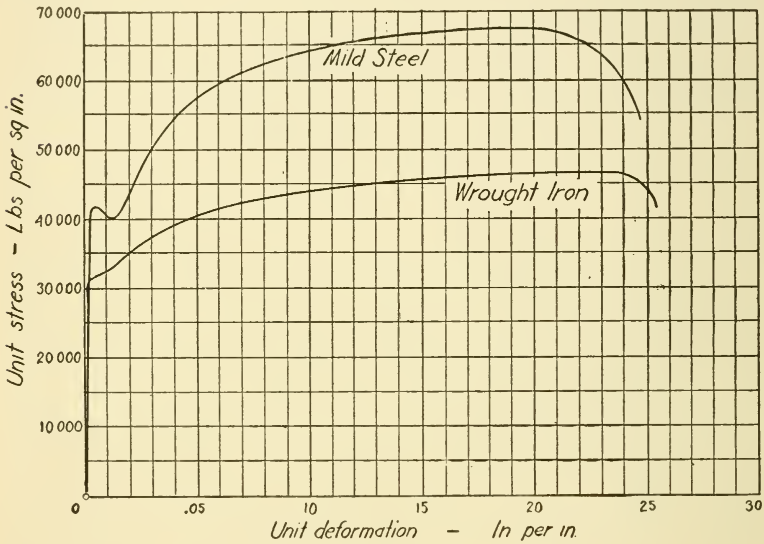


Fig. 147

stress nor as to any relation between them except that the unit stress finally reaches a maximum called the ultimate strength.

In Fig. 146*A* it will be noted that rupture occurs at a unit stress which is less than the ultimate strength. The apparent absurdity of this is removed when it is explained that the unit stress throughout is computed on the basis of the *original* cross-sectional area of the test piece. Just before rupture (in the case of ductile materials) the area reduces rapidly. If the unit stresses were computed on the basis of the actual areas, the part of the curve to the right of *d* would continue to rise. In the curves for non-ductile materials (Figs. 146B, 148 and 149) the ultimate strength and rupture occur together on the curve.

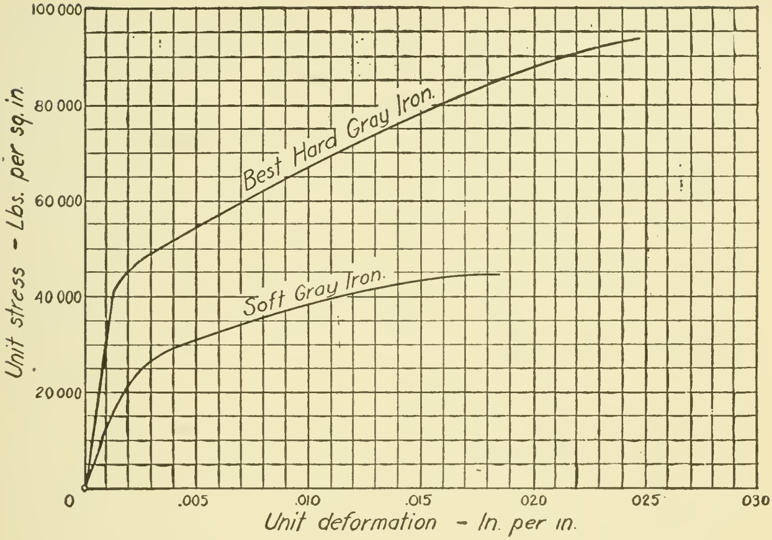


Fig. 148

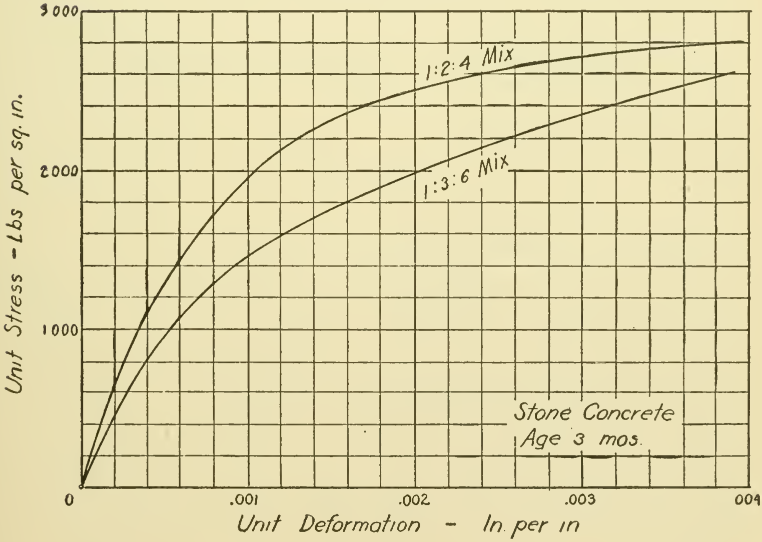


Fig. 149

69. Stress-Deformation Curves. It should be noticed particularly that the curves in Fig. 146 do not apply to any known material. Figures 147 to 149 give stress-deformation curves for a number of different materials. It will be noted that ductile materials (steel is a good example) show all the characteristics of the typical curve Fig. 146 A, while the brittle materials, such as cast iron, concrete, etc., have curves which follow the form of Fig. 146 B. In some of these latter the elastic limit is very indefinite. In none of them is there a clear-cut yield point. Often the curve terminates very abruptly, and in some cases the straight portion at the beginning is very short, the line being curved practically throughout its length.

70. The Modulus of Elasticity. The form of a typical stress-deformation curve reveals the fact that, *at all unit stresses less than the elastic limit*, there is a constant ratio between unit stress and the resultant unit deformation. This ratio is different for each different material and is called the ***modulus of elasticity*** of the material.

These facts may be expressed by the equation

$$\frac{\text{Unit Stress}}{\text{Unit Deformation}} = \text{Modulus of Elasticity},$$

or, in abbreviated form,

$$\frac{P/A}{q/L} = E. \tag{1}$$

In which q represents the total deformation.

It will be seen that when a large unit stress produces a small unit deformation, i.e., when the material has a great resistance to deformation, the value of E is large, and vice versa. Putting this in another way, the modulus of elasticity of a material is a measure of its *stiffness*. Or again, referring to Fig. 146 A, the modulus of elasticity is expressed by the slope of the straight part of the stress-deformation curve, and is equal to the tangent of the angle θ.

For an absolutely brittle material, $E = \infty = \tan 90°$,

For an absolutely ductile material, $E = 0 = \tan 0°$.

By assigning to each term in the left side of equation (1) its value in pounds, inches or square inches, it will be evident that E is expressed in pounds per square inch. This will be better understood if we imagine a case in which the deformation q is equal to the original length L. In such a case the denominator of the left side of equation (1) becomes unity and E is seen to be equal to the unit stress. In other words, E is that unit stress which would produce, in a given material, a deformation equal to its original length. Of course no structural material would allow of such deformations. The idea is a purely fanciful one, but it does shed light on the general character and significance of modulus of elasticity.

The modulus of elasticity for most materials of construction is about the same whether determined from tensile or compressive tests, except in the case of the more brittle materials. Modulus of elasticity for shear is discussed in § 72. Average values for E are given in Table I in the Appendix.

The student should particularly note that the ratio which is called E holds good only at stresses *less than* the elastic limit, since it is only in such a case that the ratio of stress to deformation is a constant quantity. No problem involving greater stresses can be solved by using the above relations. On the other hand, when the stresses involved are within the elastic limit, equation (1) can be used in any one of several ways.

As an example, suppose a piece of material 12′ 0″ long and 1″ × 1″ in cross section is found to elongate 0.0767″ under a load of 16,000 lbs., and it is required to find the modulus of elasticity of the material. Then we have

$$E = \frac{\dfrac{16{,}000 \text{ lbs.}}{1'' \times 1''}}{\dfrac{0.0767''}{144''}} = 30{,}000{,}000 \text{ lbs./sq. in.}$$

Again, suppose that the material of a bar is known to have its $E = 30{,}000{,}000$ lbs./sq. in., that its length is 12′, and that its area of cross-section is 1 sq. in. It is required to know the pull X,

which will produce an elongation of 0.0767″. Then the same relation gives

$$\frac{\dfrac{X}{1'' \times 1''}}{\dfrac{0.0767''}{144''}} = 30{,}000{,}000 \text{ lbs./sq. in.,} \qquad \text{and} \qquad X = 16{,}000 \text{ lbs.}$$

Similarly, if any four of the quantities in equation (1) are known, the fifth may be determined.

To sum up the whole matter, the modulus of elasticity is the direct measure of the stiffness of a material. For* any given material, E is a constant quantity for all cases of stress within the elastic limit. It is expressed in pounds per square inch, and its numerical value for structural materials runs up into the millions. On the stress-deformation curve for any material, E is shown by the inclination of the straight line part.

PROBLEMS

√ 1. A piece of material 1″ × 2″ and 12′ 0″ long elongates 0.006″ under a load of 2000 lbs. What is its modulus of elasticity?

2. (*a*) What will be the elongation of a bar of steel 18′ 0″ long and $1\frac{1}{2}$″ in diameter, under a load of 40,000 lbs.? (*b*) If the load is 65,000 lbs.?

√ 3. What unit stress will shorten a block of bronze, 2″ × 2″ and 6″ long, by one ten-thousandth of its length?

√ 4. How much load will cause 0.006″ elongation in a bar of aluminum, 9′ 2″ long and 2″ × $\frac{1}{2}$″ in cross section?

5. A wrought iron rod 1″ in diameter and 208′ 0″ long is fitted with a standard turnbuckle and thread. Supposing the ends of the rods to be fastened to rigid supports and the turnbuckle in the center screwed up to produce a tension of 500 lbs. in the rod. How much additional stress will be caused by the next half turn of the turnbuckle. (See handbook for details of turnbuckle and threading of rod.)

6. A load is to be supported as in Fig. 31. Each wire is of wrought iron, $\frac{3}{8}$″ in diameter. (*a*) How much load will be required to lower the ring $\frac{1}{100}$″? (*b*) How much load will be required to lower the ring $\frac{1}{4}$″?

7. An elevator plunger is of steel, 6″ in diameter outside and $5\frac{1}{2}$″ inside. It is 250′ long. What is the difference in its length when suspended and when resting on its lower end. The weight of the car is taken up by counter weights.

* Certain exceptions are found in this statement. In the case of concrete and other brittle materials the stress-deformation curve is a curved line almost from the beginning (Fig. 149). In such a case the E is evidently a variable, being different for each different unit stress.

8. If a bar of a given material is 1″ in diameter and 8′ 0″ long, and if the elongation of this bar under a load of 15,000 lbs. is found to be 0.05″, how much elongation will be produced in a bar of the same material which is 2″ in diameter, 12′ 0″ long, and carrying a load of 30,000 lbs.
9. A steel bar 50′ 0″ long is suspended from one end and hangs vertically but carries no load except its own weight. What is the total elongation?
10. A round tapered steel rod is 2″ in diameter at one end, 1″ in diameter at the other end, and 20′ long. What will be the elongation under a tensile load of 15,000 lbs.?
11. From the data given in Table V in the Appendix, deduce the elastic limit, ultimate strength, and modulus of elasticity for each case.

71. Shearing Stress. Forces producing tension or compression act in the same straight line and in opposite directions. When applied axially (§ 56), they produce stresses which are uniformly distributed over a section *at right angles to* the direction of the forces. The case of coplanar forces, oppositely directed but not in the same straight line, is more complex. It may occur in one of two cases as follows: (A) the oppositely directed forces may be well separated, (B) they may be adjacent.

Case A is illustrated by such an arrangement as is shown in Fig. 72. In such a case the member carrying the load is a beam and bends in a manner familiar to all. The stresses involved are discussed in Chapters XIII–XVIII.

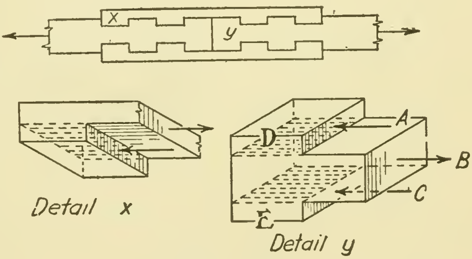


FIG. 150

Case B is illustrated by the ordinary fish-plate timber splice shown in Fig. 150. In detail y the forces A and C, being oppositely directed to B, tend to slide the blocks D and E along the main timber. This action causes stress on the two shaded areas.

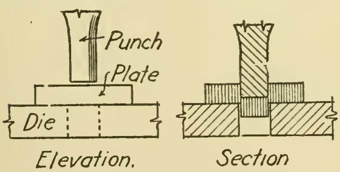


FIG. 151

Again, the same kind of stress occurs when a hole is punched in a plate, as shown in Fig. 151. When the punch descends, it causes stress due to oppositely directed forces on each side of the

periphery of the plug which is punched out of the plate. Such stress is called shearing stress. Notice that shearing stresses occur on areas extending parallel to the direction of the forces.

Obviously there can be no sharp division between cases A and B above. "Well separated" and "adjacent" are words that have no precise meaning. But in general, when the distance between the oppositely directed forces is small in comparison with the dimensions of the material under stress, the stresses are shearing rather than bending and may be considered as uniformly distributed over the areas receiving the stress.

Let it be required to determine the force required to punch a hole $\frac{3}{4}''$ in diameter in a plate $\frac{1}{2}''$ thick, the ultimate shearing strength of the metal in the plate being 40,000 lbs./sq. in.

The sheared area is $\frac{3}{4} \times 3.1416 \times \frac{1}{2} = 1.178$ sq. in. and the force required is 40,000 lbs./sq. in. $\times$ 1.178 sq. in. = 47,120 lbs.

72. Shearing Deformation. Bodies subject to shearing stresses deform quite differently from those under tension or compression, the deformations being angular rather than linear. Figure 152*A* represents a block held in a clamp and acted upon by the force *F*. Shearing stresses exist on any horizontal plane passed through the block. The top section of the block transmits the force *F* to the section below by reason of shearing resistance on the plane *AA*. And each section in turn transmits to the one below until the reaction is obtained on the clamp.

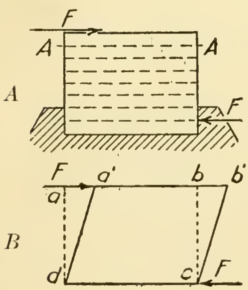


Fig. 152

The deformation produced on each plane is the same, but it accumulates toward the top, as shown in Fig. 152*B*. Such deformation is measured by the ratio between the horizontal displacement aa' and the vertical height ad; that is, it is the tangent of the angle ada'. It gives rise to a shearing modulus of elasticity, a full treatment of which may be found in more extended texts.

Since the shearing modulus is rarely used in structural work, the subject will not be developed further in this book.

PROBLEMS

1. What is the unit stress on the pins a and b, Fig. 32? The pins are of cast iron, 2″ in diameter.
2. If the pins s' and s'', Fig. 92, are each 1″ in diameter, what is the unit stress on each?
3. What force is required to punch a hole $1\frac{1}{2}$″ in diameter in a wrought iron plate $\frac{1}{2}$″ thick?
4. A very thin punch will fail in compression before it can be forced through a very thick plate. Derive an expression to show the thickest plate that can be penetrated by a given punch of circular cross section.
5. What is the thickest plate of wrought iron that can be punched by a steel punch $\frac{3}{4}$″ in diameter; the punch being of a steel which has a crushing strength of 120,000 lbs. per sq. in.?

73. Shear at Right Angles to Applied Forces. The forces shown as acting on the block in Fig. 152*B* would produce a clockwise rotation. It is therefore evident that the clamp in Fig. 152*A* must exert not only an horizontal reaction but also a counter-clockwise moment as shown by the free body in Fig. 153.

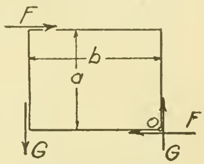


Fig. 153

The forces G which produce this moment are a part of the necessary reaction to the force F. Their amount is found by equating the moments of the two pairs about some point as o. Then

$$(1) \qquad Fa = Gb.$$

Now it is evident that the forces G produce a shear on the block in a vertical sense. In order to determine the relative intensities of the two shears, let the thickness of the block, perpendicular to the paper, be c; let the intensity of the horizontal shear be S and that of the vertical shear be S'. Then

$$F = Sbc, \quad \text{and} \quad G = S'ac.$$

Substituting these values for F and G in equation (1), we find

$$Sabc = S'bac, \quad \text{or} \quad S = S'.$$

Hence the intensities of the shears are equal.

If now we think of Fig. 153, not as a large body but as an elementary part of any body subjected to shearing stress, the same demonstration would hold good. Thus it is evident that

whenever a body is subjected to shearing forces, two sets of shearing stresses, *equal* in *intensity* and acting at 90° to one another, are set up throughout the body.

74. Tension and Compression Accompanying Shear. Let Fig. 154*A* represent the side elevation of a rectangular prism cut from the body shown in Fig. 152. Let the arrows shown represent shearing stresses on the faces of the prism. By § 73, these stresses must occur together and their intensities must be equal.

The effect of the shearing stresses, whether considered together or separately, is to deform the parallelopiped into the form shown in Fig. 154*B*. During this deformation the diagonal *ge* has been lengthened and *df* shortened. It will now be shown that these deformations indicate the presence of tensile and compressive stresses which accompany the shear.

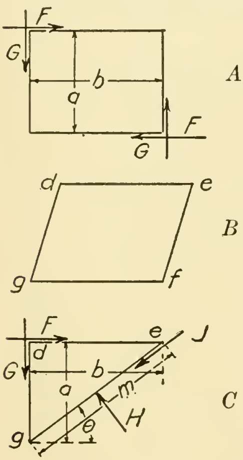


Fig. 154

Let the parallelopiped be cut by a diagonal plane and one part shown as a free body as in Fig. 154*C*. Let S represent the intensity of the shearing stresses on the faces *de* and *dg* and let c represent the depth of the parallelopiped (perpendicular to the paper). Then $F = bcS$ and $G = acS$. Now the stresses which hold the body in equilibrium may be resolved into the normal stress H and the tangential stress J. The amounts of H and J may be found by resolving F and G into components normal and parallel to *ge*. We have

$$\begin{aligned} H &= F \sin\theta + G\cos\theta, & J &= G\sin\theta + F\cos\theta,\\ &= Sbc\sin\theta + Sac\cos\theta, & &= Sac\sin\theta + Sbc\cos\theta,\\ &= 2Sbc\sin\theta; & &= Scm. \end{aligned}$$

Now if each of these expressions is divided by the area over which the respective stresses are distributed, we get

$$\text{Unit Stress due to } H = 2S\sin\theta\cos\theta = S\sin 2\theta,$$
$$\text{Unit Stress due to } J = S.$$

If we examine the first expression, we see that it becomes a maximum when $2\theta = 90°$, that is when $\theta = 45°$, and that under that condition the unit stress due to H is equal to S.

The second expression has the same value for all values of θ. Moreover if we had taken our free body by cutting a plane through the diagonal df, we would have found the normal stress on the section to be the same in amount but of opposite sign.

The above discussion, taken in connection with § 73, may be summed up as follows. When a body is subjected to shearing forces only, shearing stresses are set up on every plane passed through the body and perpendicular to the plane of the shearing forces (i.e., perpendicular to the paper in Fig. 154). The shearing stresses on all such planes are equal in intensity. There are also present tensile and compressive stresses which act normal to such planes and which reach a maximum intensity (equal to the shearing stresses) on planes at 45° to the direction of the shearing forces.

Thus if there were a material whose tensile or compressive strength were less than its shearing strength, and if shearing forces were applied, failure might result on a plane at 45° to the forces and be due to this tension or compression which accompanies the shear. (See § 78 for an illustration of this case, and § 200 for an extension of this principle.)

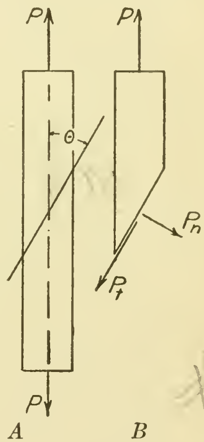


Fig. 155

75. Shear accompanying Tension or Compression. This case is analogous to that in § 74. Let the bar shown in Fig. 155A be subjected to tension as shown. The load produces a unit stress on a normal section equal to P/A. Now let a plane be passed through the bar, making an angle θ with its axis. If now one end of the bar be taken as a free body, as in Fig. 155B, it is seen that normal and tangential stresses are required for equilibrium, as shown.

The shearing stress P_t has the value $P \cos \theta$ and is distributed over an area equal to $A/\sin \theta$. Thus the shearing unit stress is

$$\frac{P}{A} \sin \theta \cos \theta = \frac{1}{2} \frac{P}{A} \sin 2\theta.$$

An examination of the expression for shearing unit stress shows that its value is zero when $\theta = 90°$ or $0°$ and a maximum, equal to one half of P/A, when $\theta = 45°$. Thus it is seen that in any material subject to tension or compression whose shearing strength is less than $\frac{1}{2}$ of its tensile or compressive strength, failure in shear on a 45°* plane may be expected. (See § 78).

In a manner similar to the above, the unit stress due to Pn may be shown to be $(P/A) \sin^2 \theta$. This value is a maximum when $\theta = 90°$, i.e., danger of failure is greatest on the cross section, as might be supposed.

PROBLEMS

1. A stick of timber, 6″ × 6″ and 10′ 0″ long, carries a tensile stress of 25,000 lbs. At one point in its length the grain is at 45° to the axis of the timber. What is the shearing unit stress parallel to the grain at that point?
2. A cube, 2″ on each side, has compressive forces of 500 lbs. acting on the horizontal faces. There are also tensile forces of equal amount on the right and left vertical faces. What is the unit shearing stress on the planes, whose traces form the diagonals of the front face?
3. A rectangular block 2″ × 3″ × 6″ is under the action of two forces of 500 lbs. each. These forces act *along the 6″ diameters* of the 2″ × 6″ faces and are oppositely directed. There is a second pair of forces acting *along the 3″ diameters* of the 2″ × 3″ faces. The block is in equilibrium. (*a*) Determine the amounts of the second pair of forces. (*b*) Determine the unit shearing stress on a plane passed parallel to the 2″ × 6″ faces and midway between them. (*c*) Determine the normal and shearing unit stresses on each of the planes whose traces form the diagonals of the 3″ × 6″ faces.
4. A steel bolt, 1″ in diameter and 12″ long, is subjected to a tension of 5000 lbs. What is the maximum unit stress in tension and in shear?
5. A brick pier is 16″ × 16″ in cross section and 6′ 0″ high. It carries a central load of 30,000 lbs. What is the maximum unit shearing stress in the pier?

* In practice this statement is modified by friction which develops on the plane of rupture.

CHAPTER IX

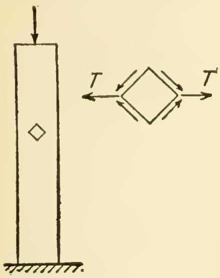
MATERIALS

76. Introduction. It is impossible, in a text of this kind, to treat completely the qualities of the materials of construction. There are many standard works devoted exclusively to this subject. But it is the purpose of this chapter to summarize the matter sufficiently to illuminate the study of that which is our chief concern, viz., the distribution of stresses and deformations in loaded structures and the proportioning of their parts in order to resist the stresses, and to minimize the deformations.

It will not be possible to show the composition and methods of manufacture nor the means and methods by which the qualities of materials are tested, measured, and correlated. It must suffice to point out that the testing of materials for each separate quality has a special technique involving usually distinct apparatus and procedure as to each step; from the taking of the specimen to the final observations. An exact knowledge of all of the detail of these operations is not essential to the structural designer. But it is essential that he should know the main elements of the composition, manufacture, and testing of materials; that he also should be familiar with the general characteristics of the usual materials and have easily available the information concerning those that are more rarely used. This of course implies the study of much more voluminous material than is presented here.

77. Elasticity. The general notions concerning the elasticity of Materials are given in Chapter VIII. There we have seen how stress is the cause of deformation: that a lengthening under tensile forces is accompanied by a reduction of cross section and a shortening under compression by an increase in section. It may be worth while, however, to point out how the ideas of §§ 74 and 75 aid in understanding the true significance of this bilateral change in size due to stress.

Consider a piece of material under compressive stress. From § 75 we know that the compression is accompanied by shearing stresses which are greatest on planes at 45° to the axis. And from § 73 we know that these shearing stresses occur in pairs in directions at 90° to one another. Conceive further an elementary cube cut from the body in Fig. 156*A* and shown enlarged in Fig. 156*B*. On its faces the shearing stresses which accompany compression are shown. These produce a resultant tension as shown by *TT* (§ 74), which acts in a direction perpendicular to the axis of the piece and which explains the increase in sectional area that accompanies the shortening in length. It will be shown also in § 78 that these tensile stresses often influence largely the manner of failure in compression.



A B

FIG. 156

The quality of elasticity (§§ 63–70) is present in most of the materials of construction to a greater or less degree when they are not stressed above the elastic limit (§ 66). Some of the metals, particularly those that are manufactured by a rolling process, like structural steel, are almost perfectly elastic. The stress-deformation diagram (Fig. 147) starts with a nearly straight line and the elastic limit and particularly the yield point (§ 67) are well defined. On the other hand, such metals as cast iron and most of the earthy materials like concrete, brick, stone, etc., have poor elastic qualities. Not only is the amount of deformation small but the elastic limit is poorly defined or non-existent. Often, in these materials, the stress-deformation curve is not a straight line (Fig. 149), and a permanent set is observed under small stresses.

If a body is permanently deformed under small stresses, it is called *plastic*. Lead, for example, when under stress starts a flow of metal under small loads. Certain grades of copper and other alloys are markedly plastic. Moreover some of the more elastic materials, like steel, become somewhat plastic under high stresses (§ 78) or high temperatures.

A body which cannot change its shape without rupture is called *brittle.* Cast iron, most of the masonry materials, and glass are the best examples, though none of them is perfectly brittle. The quality of brittleness is the reverse of plasticity.

Ductility is the quality possessed by some metals which makes it possible to draw them into wires or hammer them into sheets. During such a process the materials must undergo large deformations without failure. This quality is often confused with plasticity. It is in fact analogous. A material which is plastic under high stress, and is also tough, is ductile.

Gold is perhaps the most ductile material, as witnessed by the very thin leaf used for gilding. Soft copper sheets are sufficiently ductile to permit the stamping out of high relief ornament. Certain grades of iron are easily drawn into wire. This process of cold drawing and working often adds to the strength or raises the elastic limit of the material so treated.

78. Strength. The quality of strength is perhaps the most important to the Structural Engineer. It is apparently unrelated to the other qualities possessed by a material, though in a general sort of way heaviness and hardness, particularly the former, are considered as indications of superior strength. However, even the most superficial consideration shows, from actual cases, that there are plenty of contradictions to even so guarded a statement.

The strength of a given material may vary widely depending on the kind of stress. Thus cast iron, stone, etc., are much stronger in compression than in tension, while with wood the reverse is true. Again the strength of a material which is definitely stratified or which has a definite grain, as in the case of some kinds of stone, and all kinds of timber, may vary widely, depending on the direction of the stress, relatively to the direction of the stratification. For instance, the ultimate strength of timber in tension is forty or fifty times as great when the forces act along the grain as it is when they act perpendicular to it.

Materials which are at all plastic under high stresses show markedly different phenomena of failure than those that are more brittle. When a ductile material is stressed nearly to the

breaking point in tension, there is a marked reduction in area and finally a break which usually occurs in the form of a cup and cone. This form of break is shown in Fig. 157 which is taken

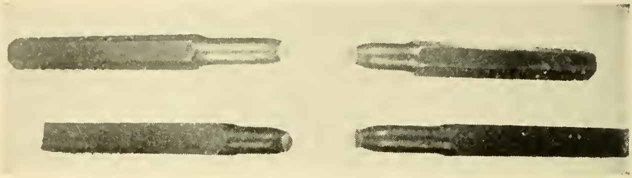


Fig. 157

from test pieces of medium steel. The cup and cone are formed approximately at an angle of 45° to the axis of the piece, as described in § 75.

If the ultimate tensile unit stress for the material is less than twice the ultimate shearing unit stress, the break will be at right angles to the axis. This form of break is shown in Fig. 158 which is taken from a piece of cast iron tested in tension.

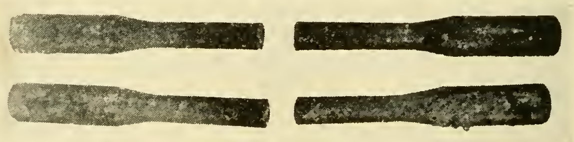
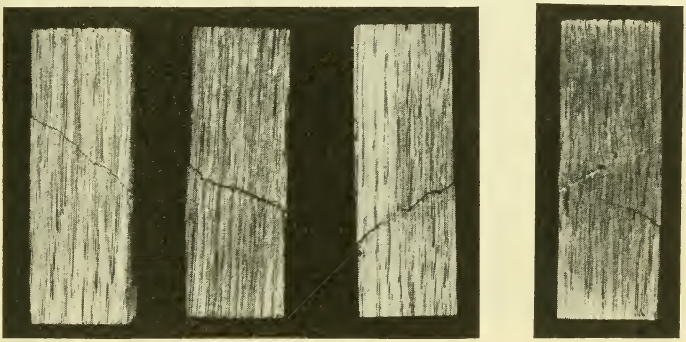


Fig. 158

Failure in compression is a very different phenomenon from failure in tension. Since the material is being forced together lengthwise, the only possible failure that can occur is in a sidewise sense. If the sides of a cube could be supported so as to absolutely prevent bulging, the ends could carry any load whatever in compression. In §§ 74 and 77 we have seen that compressive stresses produce shear on inclined planes and tension in a direction normal to that of the compression. These facts are well illustrated in Figs. 159–161. Figure 159*A* shows typical compression failures in a piece of red oak. Here the failure is due to shear, the exact angle of the plane of failure varying with the characteristics of the individual piece. Figure 159*B* shows a

piece of red oak in which failure has started on each of two planes, oppositely inclined to the axis. Figure 160 shows the typical failures in a concrete cylinder. The break occurs in a roughly



A *B*

Fig. 159

conical form due to shearing on inclined planes. Figure 161 shows typical failures in a cement cube and cylinder. In each case the double conical form of failure, due to shear and transverse tension, is quite evident.

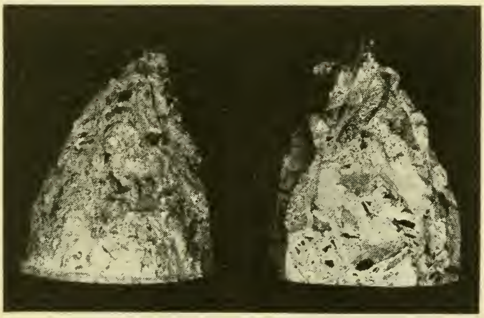


Fig. 160

Ductile materials cannot be said to have any definite ultimate strength in compression. Failure occurs, due to a gradual flow of the metal. Figure 162*A* shows a cylindrical specimen of lead, before and after it was tested in compression. The change in texture is due to the cold working which the metal undergoes

during compression. Figure 162*B* shows the end and side views of a piece of steel pipe tested in compression. The piece has been

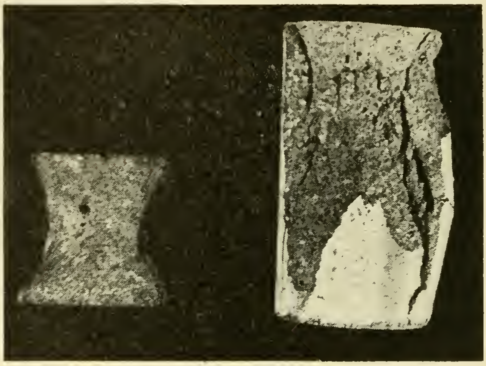
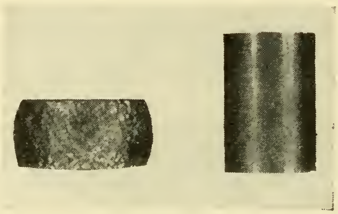


Fig. 161

distorted quite beyond recognition as a pipe but if put to a further test would show considerable strength in compression.



A

B

Fig. 162

In such cases, it is obviously impossible to set any definite ultimate strength in compression. On the other hand, there is a

definite elastic limit in compression which, in the case of wrought iron and steel, is practically the same as the elastic limit in tension.

In the case of *built up* steel *columns* the ultimate strength in pounds per square inch corresponds closely to the elastic limit for the material, as determined from small specimens (§ 183—B).

When a piece of material is so loaded in compression that the loaded area is considerably less than the entire cross-sectional area, the ultimate unit strength *of the loaded part* is considerably greater than when the load covers the entire section. This is explained readily by the fact that the unloaded part of the piece forms a restraining casing around the stressed part, thus preventing, to some extent, the sidewise movement that constitutes failure in compression. Strength values determined in this fashion are called ***bearing strength*** in order to distinguish them from the compressive strength. The amount by which the bearing exceeds the compressive strength varies with the percentage of the entire surface which is loaded as well as with the material, but general practice seems to allow about 25 per cent. increase when only 50 per cent. of the cross section is loaded.

In judging of the strength value of a material it is not wise to consider the ultimate strength alone. The elastic limit is quite as important, and the relation between the two quantities is widely different in different materials. Consequently in determining a factor of safety both the ultimate strength and elastic limit must be considered, as well as the uniformity of the material.

79. Hardness. Hardness is a property quite distinct from strength. There are a number of recognized tests for hardness which depend either on the resistance of the material to scratching or to indentation under a blow. These tests are less well standardized than those for other qualities.

Hardness may (with caution)* be taken as an indication of the resistance a material will afford against abrasion; which is of great importance in paving materials and in other places where

* Rubber is a notable exception.

wear may be expected. There are special tests for abrasion which should always be used where this factor is of prime importance.

With the same reservations as above, hardness may be taken to indicate the possible resistance to weathering.

80. Weathering. The tendency of a material to disintegrate on exposure to the weather is due to a number of causes. Water and the gases present in the atmosphere, with or without the help of sunlight, set up chemical reactions that may cause a change in color or even complete disintegration. Soluble salts in one material will often attack another material when dissolved. The electrolytic action that is set up between certain metals when brought in contact in the presence of moisture and the familiar rusting of irons are other examples.

A wholly different set of actions are those due to purely physical causes. The abrasion of rain and hail, the sliding and falling of snow and ice, and above all, the changes in size due to changes in temperature or moisture content, are, except perhaps for rusting, the most destructive forces to be encountered. Any material which absorbs much water is apt to suffer when the water freezes and, in expanding, tends to split the material.

The rotting of timbers in air, which is due to a fungus growth, and the destructive attack of various insects and borers on both timber and concrete, are cases of destructive action in a third distinct class.

81. Expansion. Nearly * all materials expand with a rise in temperature and contract as they cool off. Careful investigation has shown that the amount of this change in size is proportional to the change in temperature and also to the size of the piece. It takes place equally in all directions.

For each material there is a number (called the ***coefficient of linear expansion***) which expresses the change in length (or breadth or thickness), per degree of change in temperature, per unit of length (or breadth or thickness). These coefficients are

* Water at temperatures between 32° and 39° F. is a notable exception.

quite small decimals. Some reference books quote the coefficient as for 100° change in temperature. Every table of coefficients therefore should state clearly whether it is made up for the Centigrade or the Fahrenheit scale of temperature, and for how many degrees of change according to that scale. (See Table I, Appendix.) Coefficients of surface expansion and of volumetric expansion are also sometimes quoted but are not important for the work in hand. If needed they can be derived from the coefficient of linear expansion.

The question of expansion and contraction becomes very important when materials having different coefficients are used together. The stresses set up in such cases are discussed in § 244. It is interesting to note that materials so widely different in all other respects as are steel and concrete have practically the same coefficient of expansion. Were it not so, the use of reinforced concrete in construction could not have developed as it has.

82. Weight. The weights of materials are of great importance when dead loads are to be determined. Very full information on weights of materials is contained in all handbooks. It is important for the structural designer to know the weight per unit of volume of some of the more important materials. It is not worth while however to attempt to memorize such data but rather to gradually absorb it by use. The weight of water (which is usually taken as $62\frac{1}{2}$ lbs./cu. ft.) is one of the fundamental facts of nature and should be memorized, as it forms a sort of key fact from which other weights may be judged, and around which other facts will accumulate naturally in the mind. (See Table I.)

83. Aesthetic Qualities. The aesthetic qualities of materials form a subject concerning which there is practically no literature. Matters of color, scale, texture, and assemblage might however be made the basis of an extended treatise, for no one can hope to handle materials satisfactorily unless he adds to a knowledge of scientific tests an educated aesthetic appreciation.

84. Cost.—Availability. The questions of the cost of materials and of the available stock sizes and shapes are matters that vary

so much from time to time and from place to place as to have very little significance for the student. The practitioner perforce must keep posted on these facts as a necessary part of his equipment, but for the student they are merely an added and largely meaningless complexity. The above is not intended to imply that the student should disregard economy of design in so far as the principles are evident and universal. The fundamental principles of structural design are safety and economy. The student who fails to recognize this has missed the spirit of the entire matter. But any attempt to study prices and stocks at this stage of his development is not only useless but unwise.

85. Fatigue. It long has been known that when a piece of material is subjected to repeated loadings, failure may occur even though the unit stress at no time reaches the ultimate strength of the material as determined in the usual manner.

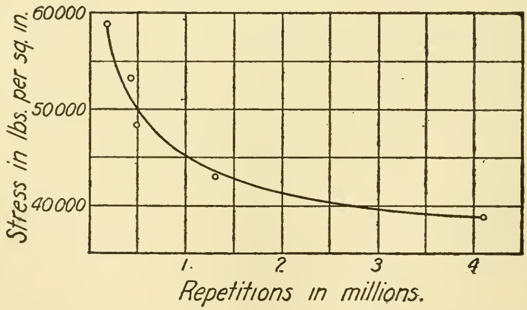


FIG. 163. Wöhler's Fatigue Tests on iron axles, under repeated bendings from zero to the stress indicated.

Experiments on this subject thus far have been largely confined to steel. Even in that restricted field, it cannot be said that the phenomenon has been explained fully. It is therefore impossible to make any statements that will apply to all materials, nor to do much more than to summarize observed facts.

From results thus far obtained it appears that:

(1) Failure may be caused by repeated loadings which cause unit stresses less than the ultimate strength of the material.

(2) With stresses varying from zero to a maximum, failure can be caused by stresses less than the elastic limit if loading is repeated sufficiently often (several millions of times). The greater the maximum load, the less will be the number of repetitions required to produce failure.

(3) With stresses varying from tension to compression, the destructive effect is greater than for either kind of stress varying from zero to a maximum.

The curve in Fig. 163 gives some concrete idea of the effects of repeated stress.

86. Steel. Steel is available in many shapes and qualities. Most of the steel used for structural purposes is rolled from hot ingots into bars, plates, and shapes of various weights and grades. The shapes and grades most generally useful have been determined by repeated trials, based on theoretical considerations. In Chapters XVI and XIX the theoretical principles governing the shapes of beams and columns are given. Other shapes, such as angles, channels and zee-bars, are based more on convenience in fabrication than on theoretical principles of strength. Familiarity with the various shapes is best acquired in practice and from the various handbooks.

Steel may be had also in the form of castings, either in stock patterns or in forms made specially for a given purpose. Steel castings if properly manufactured have about the same strength as rolled steel, but they are somewhat more brittle.

The material from which steel shapes are rolled is made in a number of grades. These grades differ from one another principally in variations in the quantity of the elements other than iron which are present in the steel; chiefly carbon, phosphorus and sulphur. The amounts of these elements which may be present and still give a steel adaptable to a given purpose have been worked out and standardized in such a way that the various grades have come to be known by the names of the purposes to which they are adapted, such as structural steel, rivet steel, boiler steel, etc.

Structural steel is a medium grade as to hardness and strength.

Its ultimate strength in tension or compression is about 60,000 lbs./sq. in., as contrasted to 50,000 lbs./sq. in., for rivet steels, and 200,000 lbs./sq. in., for cable wires. The yield point of structural steel is about one half of the ultimate strength, with the elastic limit somewhat lower. What is commercially known as high elastic limit steel is produced by cold working. In this class of material the ultimate strength is greater than for steel of the same composition that has not been cold rolled, and the yield point is nearly as high as the ultimate strength.

The modulus of elasticity is about the same for all grades of steel and is uniform at all stresses below the elastic limit,—about 30,000,000 lbs./sq. in.

Under high stress steel is quite plastic; the ultimate elongation of an 8″ piece adjoining the fracture of a test bar will average about 2″, the decrease in area at the break being around 40 per cent.

Ordinary commercial steel is about the most uniform and reliable material in common use, except in the one item of its tendency to rust. When exposure to rusting conditions is not unduly severe, factors of safety for steel may be kept fairly low.

87. Cast Iron. Cast iron is produced by running the hot metal into sand moulds. The metal has a larger carbon content than steel, and as a result iron castings are more brittle than steel.

There are several recognized grades of castings with tensile strengths varying from 12,000 lbs./sq. in. to 35,000 lbs./sq. in.; and compressive strengths between 35,000 lbs./sq. in. and 150,000 lbs./sq. in.

The elastic properties of cast iron are imperfect. The stress deformation curves in both tension and compression are more or less curved even at low stresses, and there is no definite elastic limit or yield point (Fig. 148).

Cast iron may be considered as a distinctly brittle material. Tension failures occur on a plane normal to the line of stress, and there is no visible reduction of area before failure. Compression failures occur on planes about 35° from the vertical.

From the structural point of view cast iron has two distinct uses. One is for incidental features, particularly when moulded or ornamental work is desired. Such features include stairs, elevator enclosures, frames for doors and windows, etc. The other use is for important structural members which carry compressive stress, as columns and base plates.

In either case cast iron is economical when a large number of identical parts is required, as the unit cost of the pattern is then small. However, for members carrying much stress, the use of cast iron is being discontinued in favor of steel, since its brittle quality makes it somewhat unreliable and there is a number of characteristic defects that may occur in a casting and which may remain unrecognized even after a very painstaking inspection.

The great advantages of cast iron over rolled steel are its superior rust-resisting qualities, the possibility of moulding it into any reasonable form, and, in certain cases, its cost.

88. Timber. The physical characteristics of timber vary in so many ways and over so wide a range as to be fairly bewildering. First comes a large number of species, such as oak, pine, maple, etc. Then under each of these may be half a dozen or more distinct varieties. Then, taking any given variety of a given species, the qualities of a timber will vary with the rate of growth (due to climate, exposure, soil, etc.) of the tree, and with the part of the tree from which the timber is cut. Again, the strength and other qualities of any given piece of timber are dependent to a large extent on the moisture content and upon the direction, relative to the grain, in which the stresses occur. Lastly, the presence of various natural defects, such as knots, shakes, etc., has a large effect on the strength and other qualities.

Thus (quoting from a large number of tests by the Forest Products Laboratory) the ultimate bending strength may vary from that of basswood (4,500 lbs./sq. in.) to hickory (12,700 lbs./sq. in.). Within the pine family this same quality varies from 4,700 lbs./sq. in. for sugar pine to 8,600 lbs./sq. in. for long leaf yellow pine. The rate of growth and position in the tree may, either of them, effect a variation of 20 per cent. above or below

average strength values. The presence of moisture to the extent of 20 per cent. of the dry weight may decrease the strength by as much as 40 per cent. The strength under different kinds of stress of long leaf yellow pine is about 17,000 lbs./sq. in. in tension parallel to the grain and 300 lbs./sq. in. in tension across the grain, with intermediate values for other kinds of stress. The effects of knots etc., may cause a variation of 30 per cent., more or less, either way from the average values.

It will be seen from the above that the determination of working strengths for timber is a matter requiring great care and experience, and that careful inspection is necessary unless large factors of safety are used.

The elastic properties of timber are neither as marked as in steel nor as deficient as in cast iron. The stress deformation diagram is generally straight at the beginning. There is a recognizable elastic limit at perhaps one-half of the ultimate strength, but there is usually no definite yield point. The modulus of elasticity will vary between 800,000 and 1,800,000 lbs./sq. in.

Timber is remarkably strong in proportion to its weight. This fact, taken with the ease with which it can be cut to size, assembled and worked into various shapes, makes it extremely valuable for many purposes, especially as a finishing material and for temporary structures, forms, etc. In these latter cases the ease of wrecking and the large salvage value in the timber itself are distinct advantages. On the other hand timber burns freely and is subject to decay. When exposed to the weather it must be kept painted. These qualities, together with the increasing cost of the timber itself and the labor expended on it, are at present tending to eliminate timber as a structural material for permanent buildings.

The great variation in the qualities of all wood products due to such natural defects as knots, shakes, etc., has seriously hampered the efficient use of timber for structural work. The selection, naming, and means for identifying the various grades of lumber has been so haphazard that one could never be certain of obtaining

timber of a reasonably good and consistent quality. Hence factors of safety have been kept high to cover this point. But through the combined efforts of the lumber manufacturers and the government Forest Service, grading rules are now being worked out which give promise of better conditions in this respect.

89. Brick and Stone. Brick and stone may be classed among the brittle materials. There is the same absence of well-defined elastic qualities as in cast iron. The ultimate compressive strength of individual bricks will vary between 500 lbs./sq. in. and 10,000 lbs./sq. in. while different stones will give values between 5,000 lbs./sq. in. and 25,000 lbs./sq. in. The tensile and shearing strengths are small, unreliable, and not well established. The modulus of elasticity for brick will vary between 1,000,000 and 3,000,000 pounds per square inch; and for stone between 2,000,000 and 8,000,000 lbs./sq. in.

The compressive strength of brick or stone *masonry* is much less than is that of the individual parts of which it is composed. This is due largely to the lack of uniformity throughout the mass. Because of inequalities in the mortar beds and the roughness of the individual pieces, stresses are apt to become localized and hence more destructive. Again a tight bearing may cause a brick or a piece of stone within the mass of the masonry to act as a beam rather than in compression. In such a case the low tensile strength is the initial cause of failure.

Heaviness, resistance to abrasion, and a low percentage of absorption of water are the best general guides to quality in these materials.

90. Concrete. Concrete, being a mixture of cement, sand, and a coarse aggregate with water, and being usually made on the job where careful supervision of plant and labor is difficult, is perhaps the subject of greater variation and greater abuses than any of the other materials. Each of the elements (five in all, including labor) may vary both as to quantity and quality. Again the conditions of pouring, and of curing, and the forms in which it is poured all have a vital effect on the quality of the product.

Well-made concrete at the age of 28 days should show a compressive strength of 2,000 lbs./sq. in.; a tensile strength of about $\frac{1}{10}$ as much, and a shearing strength of about $\frac{1}{2}$ of the compressive strength.

The elastic qualities of concrete are ill defined, being much the same as brick, stone, and cast iron (Fig. 149).

Curiously enough concrete has about the same coefficient of expansion as steel. This makes it possible to cast the concrete around steel bars, making what is known as reinforced concrete. When concrete has become thoroughly set around a steel bar it is very difficult to withdraw the bar. This is due to a sort of adhesion called ***bond.***

Bond is due in part to adhesion in the ordinary sense and in part to the fact that the steel bar, not being perfectly smooth and straight, is enmeshed in the concrete. The ultimate bond strength between ordinary commercial steel bars and a good grade of concrete may be taken at about 400 pounds per square inch of surface of the bar in contact with the concrete. This means that a round bar embedded in concrete a distance of about 40 diameters will develop a bond strength equal to the tensile strength of the bar.

CHAPTER X

INVESTIGATION, SAFE LOAD, AND DESIGN

91. Introduction. Most of the problems that arise in structural engineering occur in one of three forms.

(1) To determine the degree of security (factor of safety) that exists in a structure of known material and dimensions (or in some part of such a structure), under the action of a given load or loads. This operation is sometimes called "*investigation.*"

(2) To determine the *safe load* on a structure of known material and dimensions. In such a case the factor of safety may be given or it may have to be assumed. In practice it often happens that building laws or design specifications fix the working unit stresses. This amounts to the same thing as fixing a factor of safety.

(3) To *design* a structure, i.e., to determine the material and the size and shape of the parts. There are a number of variations to this problem. The material may or may not be given. The working unit stresses also may or may not be given. Again the loading may be given or merely the conditions of operation. In its most general form this problem makes the greatest call on the ingenuity and resource of the student. In that form, nothing is given except the general dimensions and purpose of the structure. Materials, loads, factors of safety, all must be the natural outgrowth of size and purpose.

92. Investigation. The size, shape, and material composing the structure or part, as well as the load placed upon it, are known, and it is required to determine (1) the existing factor of safety and (2) the sufficiency of that factor for the purposes in hand.

As an example let it be required to investigate the safety of a $\frac{1}{2}''$ diameter steel rod supporting a quiescent load of 5,000 lbs. The area of the cross section is 0.1963 sq. in. The actual unit stress is then $5{,}000 \div 0.1963 = 25{,}600$ lbs. per sq. in. The

factor of safety is 60,000 ÷ 25,600 = 2.34. By referring to Table II, it is seen that this may hardly be considered as satisfactory for most purposes.

93. Safe Load. Let it be required to find the safe load on the rod mentioned above. The ultimate strength is 60,000 × 0.1963 = 11,750 lbs. Let a factor of safety of 4 be taken as adequate. Then the safe load is 11,750 ÷ 4 = 2,940 lbs.

94. Design. Let it be required to design a thin support to carry, from above, a load of 5,000 lbs. The requirement of a thin support, carrying a load in tension, indicates the choice of a material of great tensile strength. Let us choose steel. Using a factor of safety of 4, the working strength is 60,000 ÷ 4 = 15,000 lbs. per sq. in. The required cross-sectional area is then 5,000 ÷ 15,000 = 0.333 sq. in. This corresponds to a diameter of 0.652″. The nearest commercial size *over* the required area is a round rod of $\frac{11}{16}$″ diameter.

95. Calculations. Problems of the type outlined above make a continual call on the judgment as well as the facility of the student. At this point a review of §§ 2–4 may be found worth while. No set of rules governing the approach and attack on a problem can be given. Each person has to work out his own methods, and in evaluating those methods the test of reasonableness is about the ultimate test.

PROBLEMS

1. If the pier shown in Fig. 136 is of stone masonry, what is the factor of safety?
2. A cast iron base plate (similar to Fig. 141) is 2′ 0″ × 2′ 0″ at its lower face. It rests on a concrete pier which is 4′ 0″ × 4′ 0″ on its upper face. What is the safe load?
3. A wrought iron rod, $1\frac{1}{4}$″ in diameter and 14′ 0″ long, has a hole $\frac{1}{4}$″ in diameter and 4′ 0″ long extending along the longitudinal axis of the rod from its lower end. Near the lower end a hole $\frac{1}{4}$″ in diameter is bored through the rod, perpendicular to the longitudinal axis and intersecting it. Near the upper end of the rod a square hole is cut through in a similar manner. This hole is $\frac{1}{4}$″ on each side; a diagonal of the square being vertical. What is the safe tensile load on the rod?

4. The block in Fig. 164 is made of brass. Investigate the safety of the block.
5. If the pins supporting the shear legs in Fig. 32 are of cast iron, 1″ in diameter, and the structure is intended for hoisting loads, what load may be safely hoisted? Assume that the other parts are stronger than the pins and that the pins project toward you from a solid casting.
6. Design a pin for Fig. 32 so that the structure may safely lift a load of 10,000 lbs.
7. Figure 165 represents a bolt head and washer carrying a load. Determine the necessary dimensions for (1) the diameter of the rod; (2) the height

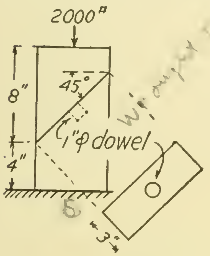


Fig. 164

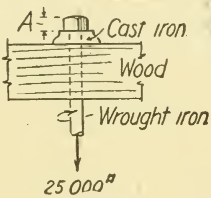


Fig. 165

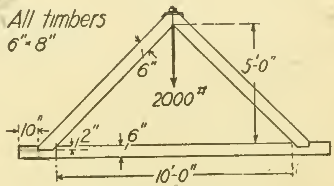


Fig. 166

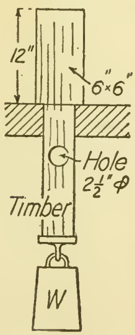


Fig. 167

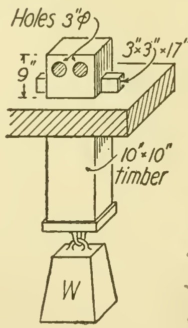


Fig. 168

of bolt head "A"; (3) diameter of the washer. Let the factor of safety be 4 on the basis of the ultimate strength.
8. Determine the factors of safety in the various parts of the frame shown in Fig. 166.
9. What is the safe load W, in Fig. 167?
10. How great a load may the timber in Fig. 168 carry with safety?

96. General Principles of Riveted Joints. The study of the stresses in a riveted joint makes an excellent application of the principles of investigation, safe load, and design to parts which are in tension, compression, and shear.

Figure 169 illustrates the simplest case of riveting. The member 1–1, carrying the tensile load XX, is discontinuous at c. In order to splice the two parts of 1–1 the plates 2–2 are placed over the joint c, matching holes having been provided previously

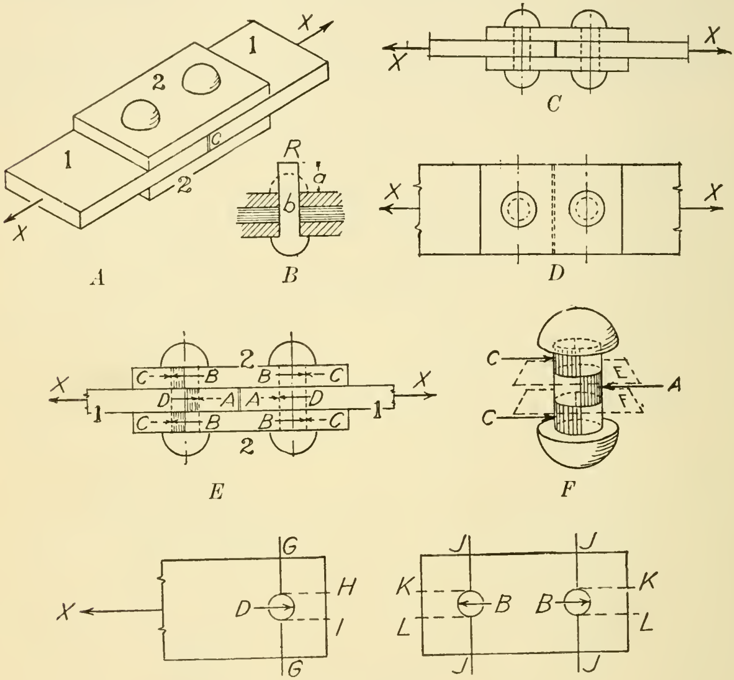


Fig. 169

in all the plates. Hot rivets R are inserted in the holes as shown in B and the over length a is hammered down into a second head (shown dotted). The forming of the head expands the shank b until it completely fills the hole. While forming the heads the plates are kept pressed together and this pressure is maintained or increased as the rivets cool. Thus there is set up a considerable friction between the plates which alone would resist a considerable pull on the joint. It is not customary, however, to consider this friction in designing joints, except as it may enter into the determination of allowable unit stresses, based on test results.

97. Stresses in a Riveted Joint. In Fig. 169*A*, *C*, and *D*, the action of the forces *X* tends to separate the pieces 1–1. As the pieces 1–1 tend to move apart, they produce pressure on the backs of the rivets, as shown by the arrows *A*, Fig. 169*E*. The rivets in turn press against the plates 2–2, as shown by the arrows *B*; these opposite pressures on plates 2–2 produce tension in them. The reactions to these pressures are shown by the arrows *C* (representing the pressure of 2–2 on the rivets) and *D* (representing the pressure of the rivets on 1–1); the forces acting *on the rivets* (*A* and *C*) are shown dotted; those acting on the plates (*B* and *D*) are shown solid.

One of the rivets is shown as a free body in Fig. 169*F*. This rivet, being short in comparison to its diameter (§ 71), is subject to shearing stresses on the planes *E* and *F*. Also it is subject to compression on the semi-cylindrical surfaces shown shaded.

One of the main plates 1 is shown free in Fig. 169*G*. The forces *D* and *X* produce tension on the minimum section *GG*. Also the force *D* produces compression on the semi-cylindrical surface of the rivet hole. Again the force *D* tends to push out the strip of metal behind it (shown dotted), causing shearing stresses along the sections *H* and *I*.

One of the splice plates 2 is shown in Fig. 169*H*. It is under the same kinds of stress as 1; tension on *JJ*; compression on the sides of the rivet holes and shear on *K* and *L*.

It will be well to note that, because of the way the rivet fills the holes and the way the heads cover the metal surrounding the holes, the compressive stresses set up where the rivets bear against the sides of the holes will classify as bearing stresses (§ 78) rather than compressive stresses. Figures 170 to 173 are from photographs of riveted joints which have been tested to destruction.* These illustrations are taken from a series of joints which were designed to have equal strengths in tension, compression, and shear. Hence, quite naturally, each of the

* These photographs were taken from a series of tests made in the laboratories of the College of Engineering at Cornell University under the direction of Professors E. N. Burrows and H. H. Schofield.

typical forms of failure is found in the series and in some cases failure in two or more ways can be recognized in the same joint.

Fig. 170

Fig. 171

Fig. 172

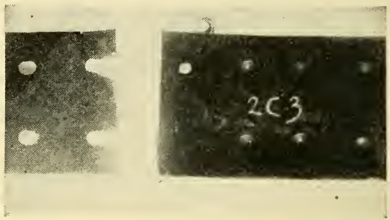


Fig. 173

Figure 170 shows a tension failure in the outside plates of a butt joint with two splice plates. Figure 171 shows a shearing failure in a butt joint with one splice plate. The rivet shows the effects of the eccentric loading as well as of direct shear. In Fig. 172 the metal behind the rivet has failed and the rivets were pulled out endwise. Figure 173 shows at the left the typical failure in compression. The holes in the plate have become elongated by excessive pressure. In the same joint there is evidence of shearing failure and pull out.

Before proceeding to follow the numerical determinations in §§ 98 to 100 the student should be sure to thoroughly comprehend the various stresses in the joint described above and the areas on which they occur. The problem of the riveted joint is primarily one of complete visualization.

98. Investigation. The same joint as was used in § 97 is redrawn in Fig. 174, with dimensions added. Let us assume the ultimate strength of the metal to be 60,000 lbs./sq. in. in tension,

45,000 lbs./sq. in. in shear, and 80,000 lbs./sq. in. in bearing (§ 97). Let it be required to investigate the joint.

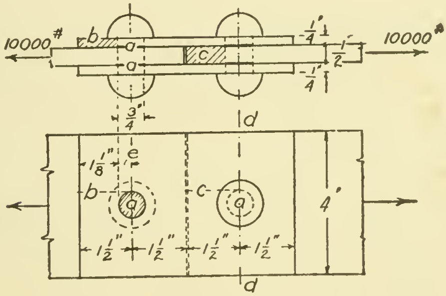


FIG. 174

Shearing of the rivet will occur, if at all, on the areas a; and the unit stress developed will be

$$10{,}000 \div 2[3.1416 \times (\tfrac{3}{8})^2] = 11{,}320 \text{ lbs./sq. in.},$$

and the factor of safety will be

$$45{,}000 \div 11{,}320 = 4 \text{ (about)}.$$

Shearing of the plate will occur, if at all, on the areas b (in the cover plate) or c (in the main plates). There are four areas b and two areas c on each side of the joint, but since the thickness of each cover plate is half that of the main plates, the total area in either case is the same. Since these areas come tangent to the rivet holes, there is some chance that failure might occur along a diagonal line from some part of the surface e to the hole. Consequently the length of these surfaces is taken as only $1\frac{1}{8}''$, to be on the safe side. The unit stress developed on the areas b (or c) will be

$$10{,}000 \div 4(\tfrac{1}{4} \times 1\tfrac{1}{8}) = 8{,}888 \text{ lbs./sq. in.},$$

and the factor of safety will be

$$45{,}000 \div 8{,}888 = 5 +.$$

Tension on the plates will cause failure, if at all, on the section d. The cover plates having, together, the same thickness as the

main plates, the unit stresses in all will be the same and equal to

$$10{,}000 \div \tfrac{1}{2}(4 - \tfrac{7}{8}) = 6{,}400 \text{ lbs./sq. in.,*}$$

and the factor of safety will be

$$60{,}000 \div 6{,}400 = 9.4.$$

Bearing will occur on the semi-cylindrical areas forming contact between the rivet shank and the plates. These areas are the same whether figured for the cover plates or the main plates.

Pressure between convex and concave surfaces doubtless gives rise to different deformation and hence to a different distribution of stress than is the case when plane surfaces are pressed together. But the exact manner in which these stresses act has not been satisfactorily determined. However repeated tests show that the effect produced is the same as *would be* produced *if* the applied force acted on a plane surface equal in area to the projection of the cylindrical surface on a plane. That is, the surface in contact between the rivet and the plate is treated *as if* it were a plane surface as long as the diameter of the rivet and as wide as the thickness of the plate.

With this in mind one can now investigate the bearing on the joint as follows:

$$10{,}000 \div 2(\tfrac{1}{4} \times \tfrac{3}{4}) = 26{,}666 \text{ lbs./sq. in.,}$$

and the factor of safety will be

$$80{,}000 \div 26{,}666 = 3.$$

The degree of security of this joint is evidently limited by its bearing strength, and the factor of safety of the joint is 3.

99. Safe Load. Let it be required to determine the safe load on the joint shown in Fig. 175. Let the ultimate strength of the metal be the same as in § 98 and let a factor of safety be 4. The

* The hole usually made to receive a $\frac{3}{4}''$ rivet is $\frac{13}{16}''$ in diameter. These holes are usually punched and the punching injures the metal immediately around the hole. Therefore it is usual in figuring net areas to deduct $\frac{1}{8}''$ more than the diameter of the rivet from the gross width of the plate.

shearing strength of the rivets is *

$$4 \times 2 \times 0.3068 \times \frac{45{,}000}{4} = 27{,}600 \text{ lbs.}$$

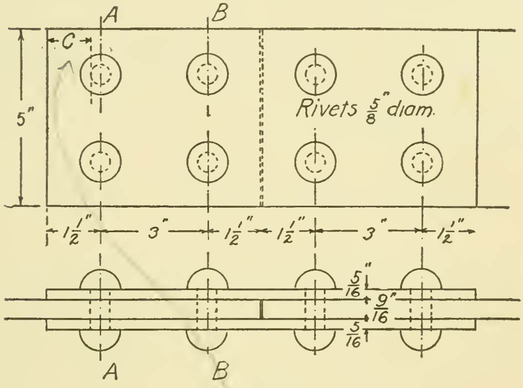


FIG. 175

The shearing strength of the plates is determined by that of the main plates, which are thinner than the combined cover plates:

$$4 \times 2 \times 0.5625 \times 1.1875\,\dagger \times \frac{45{,}000}{4} = 60{,}100 \text{ lbs.}$$

* For such computations as these it will generally be found better to convert all fractions into their decimal equivalents.

† The length of this shearing surface is figured as indicated by C (Fig. 175). The shearing surfaces between the rivets are longer; but on the assumption that all four rivets will carry equal loads, the safety of the entire joint is predicated on the shorter surfaces.

As a matter of fact the distances between rivets and from any rivet to the edge of a plate are usually determined from rules which are intended primarily to care for stresses developed in punching the holes and for the clearances demanded by punching and riveting machinery.

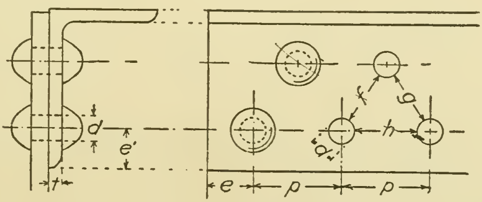


FIG. 176

Like all empirical rules, the rules for rivet spacing vary somewhat. In Fig. 176 the principal rules are illustrated. When d = diameter of the rivet, then the

The *tensile strength* of the joint will be determined by the section of the main plates taken through AA. The tensile stress developed on this section of the main plates is twice that on section BB. The reverse is true in the cover plates. We find

$$0.5625 \times (5 - 2(0.75)) \times \frac{60{,}000}{4} = 29{,}500,^*$$

and the *bearing strength* is

$$4(0.5625 \times 0.625) \times \frac{80{,}000}{4} = 28{,}100 \text{ lbs.}$$

The safe load on the joint is seen to be limited by the shearing strength of the rivets and to be equal to 27,600 lbs.

100. Design. Let it be required to design a joint similar to Fig. 175, to carry a load of 40,000 lbs., using the unit stresses given in § 98, and a factor of safety of 4. The problem admits of many solutions as the size and the number of rivets employed are interdependent. It is usual to start by choosing a definite diameter for the rivets. This choice ordinarily depends on the general character of the work of which the riveted joint is a detail. Let us assume that $\frac{5}{8}''$ rivets will be appropriate and proceed to determine the other details on that basis.

The shearing strength of one rivet is then

$$2 \times \frac{3.1416 \times (0.625)^2}{4} \times \frac{45{,}000}{4} = 6{,}900 \text{ lbs.},$$

and the number of rivets required on each side of the joint will be $40{,}000 \div 6{,}900 = 5 +$, or 6 rivets. The thickness of the plates should be sufficient so that the bearing strength of each rivet is at least equal to its shearing strength. In this case let t = the

pitch p = about $3d$. The distance to a rolled edge e' = about $1\frac{1}{2}d$; and to a sheared edge e = about $1\frac{3}{4}d$. The maximum thickness of the plates $t = d$. When rivets are staggered, $f + g \geqq h$. For boiler work the standard rule is that $f + g$ must be 20% greater than h.

When the above rules are followed the shearing resistance behind or between rivet holes will exceed the bearing strength of the rivet (Let the student check this). Hereafter these spacing rules will be followed and the computation of these shearing stresses will be omitted.

* See footnote, page 130.

thickness of the main plates. Then we have

$$6{,}900 = \frac{80{,}000}{4} \times 0.625t, \qquad \text{or} \qquad t = 0.55''.$$

Since the cover plates must each take one half as much stress as the main plates, either in bearing or in tension, their thickness should be one half of the above, or 0.275″. The minimum net width of plates required to give sufficient tensile strength is

$$40{,}000 \div \left(\frac{60{,}000}{4} \times 0.55\right) = 4.85''.$$

The gross width of the plates will depend on the way the rivets are arranged. If we use two rivets in the width of the plate, as in Fig. 175, the gross width is $4.85 + 2(0.75)$* = 6.35″. As a final detail the above decimal results should be converted into the equal or next greater fractions, since this class of work is usually executed in fractional measurements. This joint is shown in Fig. 177.

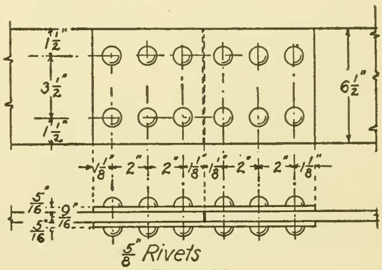


FIG. 177

A different arrangement for the same joint is shown in Fig. 178. In this joint the full 40,000 lbs. of stress is carried on the main plates at section AA, where only one rivet hole is to be deducted. Hence this plate could be made $\frac{3}{4}''$ narrower than the one in Fig. 177, except for the fact that the punching rules require more width for the rivets in the back row. The cover plates carry the full stress on section CC, and hence would need to be thickened slightly. Where long main plates carrying large stresses are to be spliced, this arrangement is often very serviceable.

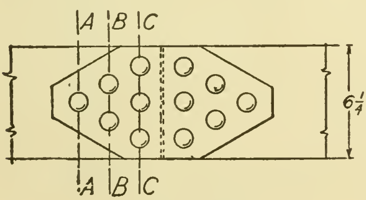


FIG. 178

* See footnote, p. 130.

101. Other Types of Riveted Joints. In Fig. 179*A* is shown a simple lapped joint. In this case the rivets tend to shear on one plane only and are said to act in single shear. The rivets in Fig. 169 are in double shear. A lapped joint is less desirable than a butt joint (Fig. 169), in that the forces are out of line and tend to bend the plates so as to bring the forces in line, as shown in Fig. 179*B*. This bending of the plates also brings tensile stresses on the rivet heads.

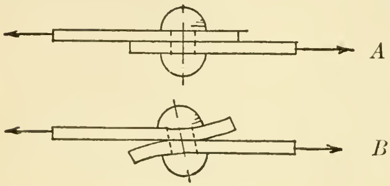


FIG. 179

Such joints are sometimes designed with smaller working stresses than are ordinarily used for joints where the loading is not eccentric.

Figure 180 shows two typical structural connections. In (*a*) the rivets *c* are in double shear and *d* in single shear. The bearing strength is determined by the thickness of the angles,

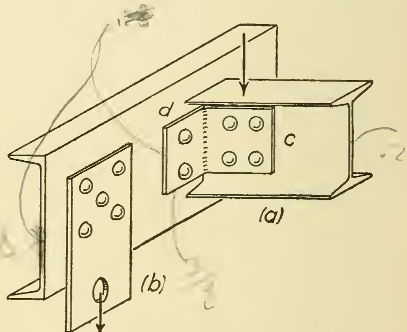


FIG. 180

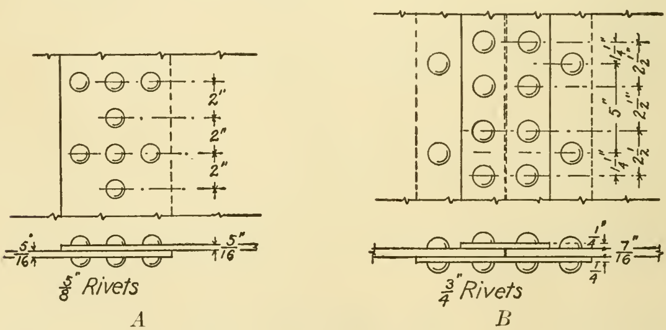


FIG. 181

or by the webs of the beams, whichever happens to be thinner. At (*b*) is shown a typical hanger connection with rivets in single shear. Figures 181*A* and 181*B* show other types of riveting commonly used in boiler work. In such work the joints are

usually very long, and it is important to devise a joint which decreases the tensile strength of the unpunched plate as little as possible. The efficiency of such a joint is measured by the tensile strength of the joint divided by that of the unpunched plate, expressed as a percentage. In figuring the strength of such a joint, it is usual to deal with a unit width, based on the rivet spacings.

PROBLEMS

NOTE. When not otherwise specified, the riveted joints in the following problems will be assumed to be of steel.

1. What is the safe load on the riveted joint shown in Fig. 182*A*?
2. Investigate the riveted joint shown in Fig. 182*B* for a tensile stress of 25,000 lbs. Side elevation same for both *A* and *B*.

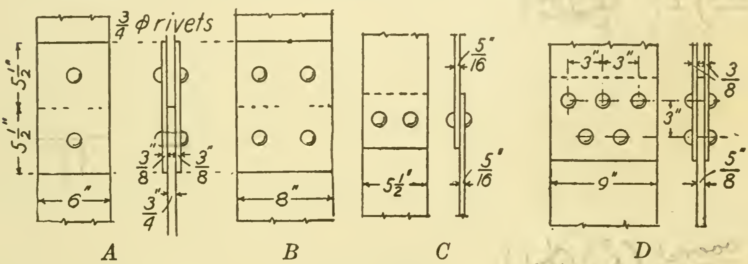


FIG. 182

3. What is the safe load on the riveted joint, Fig. 182*C*? Rivets are $\frac{5}{8}''$ in diameter.
4. If the joint in Fig. 182*D* must carry 15,000 lbs., what must be the size of the rivets?
5. In Fig. 182*D*, if the joint is to carry 3,000 lbs., what is the necessary diameter for the rivets? Let all parts be of aluminum.
6. Design a riveted joint similar to Fig. 178 to carry a tensile load of 100,000 lbs. Let the rivets be $\frac{3}{4}''$ in diameter.
7. In Fig. 180 (*b*), let the channel be $12'' \times 20.5$ lbs. and let the plate be 8″ wide and $\frac{3}{8}''$ thick. The rivets are $\frac{3}{4}''$ in diameter. What is the safe load?
8. In Fig. 180(*a*), the rivets are $\frac{3}{4}''$ in diameter. The channel is $12'' \times 20.5$ lbs. The beam is $8'' \times 17.5$ lbs. Find the greatest allowable reaction for the beam.
9. What is the strength of the riveted joint in Fig. 181*A*, per foot of width of the spliced plates?
10. What is the strength of the riveted joint in Fig. 181*B*, per foot of width of the spliced plates?
11. If the riveted joint shown in Fig. 177 is made of brass throughout, what is the safe load?
12. In Fig. 182*D*, if the plates are of brass and if the rivets are of copper, 1″ in diameter, what is the ultimate strength of the joint?

CHAPTER XI

UNIFORMLY VARYING FORCES AND STRESSES

102. Introduction. If all the loads which come on a structure were concentrated at definite points or were uniformly spread over definite areas, and if all members could be placed so that the loads would produce simple axial stresses (§ 56), the design of structural parts would be a comparatively simple matter. But, in reality, many cases occur in which the loads are variable or eccentric or the resultant stresses are not uniformly distributed over the cross sections, or both.

Such problems occur in many forms, some of which admit of fairly simple solutions, while in others the solutions are more complex. In every case, the laws of equilibrium (§ 32) and the free body method (§ 23) furnish the basis of attack on the problem, though the form of the equations set up and the mathematical solutions needed may vary considerably.

One such problem in a slightly disguised form already has been noticed (§ 52—C). In that case the elementary area strips are treated as uniformly varying forces. In this chapter we shall take up some typical problems of this class, and, having solved them, we shall deduce a general principle for all such cases.

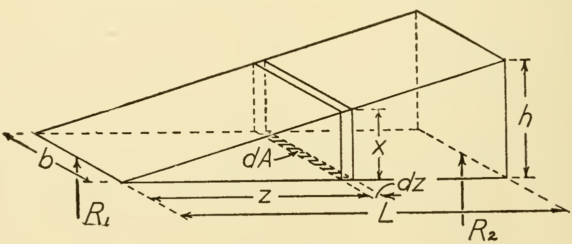


Fig. 183

103. Reactions of Beams with Uniformly Varying Loads. A. Beam with Rectangular Plan. Let the beam shown in Fig. 183 be made of a material having a heaviness of g lbs. per cu. in.

Let the weight of the beam be W lbs. and let the (unknown) moment of this weight, taken about R_1 be expressed by M. Let it be required to determine the reactions (R_1 and R_2) due to the weight of the beam. From the relation $\Sigma V = 0$ we get

(1) $$R_1 + R_2 = W,$$

and, taking moments about R_1, from $\Sigma M = 0$ we get

(2) $$R_2 L = M.$$

These two relations furnish the basis for the solution. In order to get a definite value for M in equation (2) above, let an elementary strip of the beam be chosen, as shown in the figure, at the distance z from R_1. Let the plan area of the strip be dA. Its volume is $x\,dA$, and its weight is $gx\,dA$. The moment of the weight of the strip (about R_1 as a center) is then $zgx\,dA$. Therefore the moment of the entire weight of the beam is *

(3) $$M = \int_A zgx\,dA.$$

Now from similar triangles, we have

$$x : h = z : L, \qquad \text{or} \qquad x = z\frac{h}{L}\cdot$$

Substituting this value of x in equation (3), we get

(4) The moment of the varying load $= \int_A zgz\frac{h}{L}\,dA = g\frac{h}{L}\int_A z^2\,dA.$

Now from equations (2) and (4), we have

(5) $$R_2 L = g\frac{h}{L}\int_A z^2\,dA.$$

But $dA = bdz$; substituting this value in equation (5), we find

(6) $$R_2 L = bg\frac{h}{L}\int_A z^2\,dz.$$

In this equation z may have any value from zero to L; and when

* See footnote, p. 63.

the above expression is made definite and integrated for these limits, we obtain

$$R_2L = bg\frac{h}{L}\frac{1}{3}L^3 = \frac{1}{3}bghL^2,$$

whence

(7) $$R_2 = \tfrac{1}{3}bhgL.$$

The weight of the beam W as used in equation (1) can be expressed by $\frac{1}{2}bhgL$. Therefore equation (1) may be written in the form

(8) $$R_1 + R_2 = \tfrac{1}{2}bhgL.$$

Substituting the value of R_2 from (7) in (8), we get

$$R_1 = \tfrac{1}{6}bhgL.$$

In other words R_1 = one third of the weight of the beam, or $\frac{1}{2}R_2$.*

B. Beam with Semi-Circular Plan. Figure 184 represents a semi-circular plate supporting a load which varies uniformly from zero at AB to 150 lbs. per sq. ft. at C. Let it be required to determine the supporting reactions.

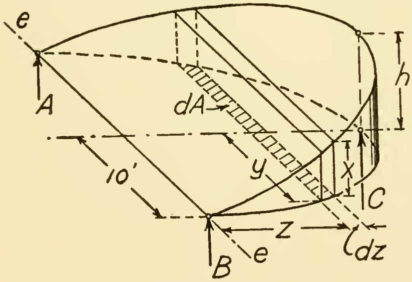


Fig. 184

From symmetry and the conditions of equilibrium, we have

(1) $A = B$,

(2) $A + B + C$ = total load,

(3) $10C$ = moment of total load about e–e.

Let an elementary strip of the plate be chosen, as shown by dA. The pressure per sq. ft. on this area is to 150 lbs. (the pressure per sq. ft. at C) as x is to h. But from similar triangles

(4) $$\frac{x}{h} = \frac{z}{10}.$$

* This solution is practically a repetition of the one in § 52—C. It can be very much simplified by using the known position of the center of gravity of a triangle.

Therefore, the pressure per sq. ft. on $dA = 150(x/h) = 15z$. Then the total pressure on $dA = 15z\,dA$, and the moment of this pressure, about e–e, $= 15z^2\,dA$. From this it follows that the entire load will produce a moment, about e–e, which can be expressed as

$$\int_A 15z^2\,dA. \tag{5}$$

Now we further know that $dA = 2y\,dz$. Also, from the properties of a circle, we have

$$z^2 + y^2 = 100, \qquad \text{or} \qquad y = (100 - z^2)^{1/2},$$

whence

$$dA = 2(100 - z^2)^{1/2}dz.$$

Substituting this value in equation (5), we find that the moment M of the entire load is

$$M = 15\int_0^{10} 2z^2(100 - z^2)^{1/2}dz = 30\int_0^{10} z^2(100 - z^2)^{1/2}dz.$$

Evaluating this, we have

$$M = 30\left[\frac{z}{4}\sqrt{(100 - z^2)^3} + \frac{100}{8}\left(z\sqrt{100 - z^2} + 100\sin^{-1}\frac{z}{10}\right)\right]_0^{10}.$$

When this is evaluated, the moment is found to be 58,900 lbs. ft. Using this value in equation (3), we get the value of the reaction C as 5,890 lbs.

The reactions A and B may now be determined. First let us determine the entire load on the plate. The load on the elementary strip is $15z\,dA$ as shown above. Also $dA = 2y\,dz$, and $y = (100 - z^2)^{1/2}$, all as shown above. Therefore the load on the elementary strip is

$$(15z)(2(100 - z^2)^{1/2}dz) = 30z(100 - z^2)^{1/2}dz.$$

The entire load on the plate is then

$$\int_0^{10} 30z(100 - z^2)^{1/2}dz.$$

Evaluating this expression, we find that the entire load is 10,000

lbs. Then, from equations (1) and (2), it can be shown that $A = B = 2{,}055$ lbs.

104. Hydraulic Pressure on a Valve Plate. In Fig. 185 is shown a dam with rectangular opening at the bottom blocked by a single hinged valve. Let it be required to find the force P required to keep the valve closed.

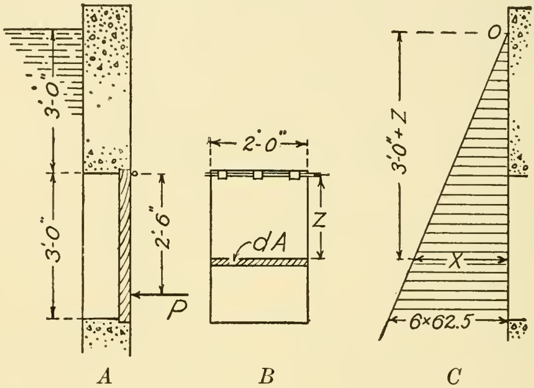


Fig. 185

Let dA be an elementary strip of area on the face of the valve, and let x be the intensity of the water pressure on this strip. From the law of hydraulic pressure illustrated in Fig. 185C; $x = 62.5(3 + z)$. Then the total pressure on the elementary strip is $62.5(3 + z)dA$. The moment of this pressure about the hinge will then be

(1) $$z62.5(3 + z)dA.$$

But $dA = 2dz$. Substituting this in (1), we find

$$\text{Moment of pressure on elementary strip} = 125z(3 + z)dz.$$

(2) $$\text{Total moment of pressure on valve} = 125\int_0^3 (3z + z^2)dz,$$
$$= 2{,}812 \text{ lb. ft.}$$

Now the moment of P about the same center $= 2.5P$. Hence

$$2.5P = 2{,}812, \qquad \text{or} \qquad P = 1{,}125 \text{ lbs.}$$

105. Eccentric Load on Rectangular Block.* Figure 186 shows a short block, rectangular in plan and resting on a plane surface $M - M$. It carries a load P which is on one axis of the top surface but not on the other.* In such a case, the stresses on any

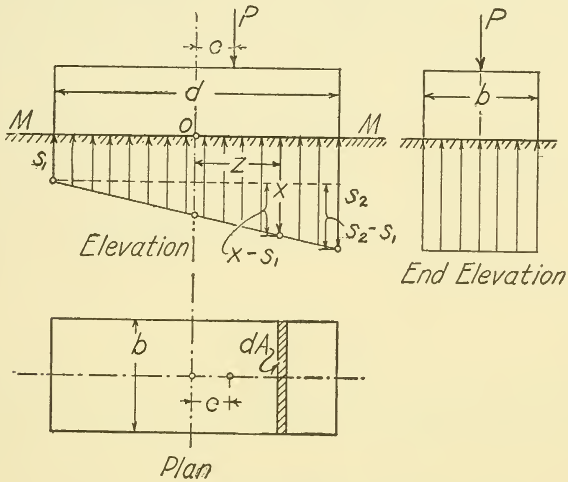


Fig. 186

cross section, as well as the reactions underneath the block are not evenly distributed over the cross section as in the case of axial loads. Investigation has shown that the unit stresses under the block vary *uniformly* from a maximum at one edge (s_2 in Fig. 186) to a minimum at the other (s_1), provided that the block is relatively stiff and that the unit stresses are less than the elastic limits of the materials.

Let it be required to determine the unit stresses s_1 and s_2, assuming that the load P, its eccentricity e, and the dimensions of the block b and d are known. From $\Sigma V = 0$, we know that the load and the total reaction are equal, that is,

$$P = \frac{s_1 + s_2}{2} bd. \tag{1}$$

From $\Sigma M = 0$, we know that the moment of the load about any center, as o, is equal to the moment of the reaction about the

* The case of a load on neither axis is discussed in § 193, and in Chapter XXII.

same center. In order to evaluate the latter moment, choose an elementary strip of the plan dA, as shown. The unit stress on this strip is x. Therefore the total force acting on dA is $x\,dA$, and its moment about o is $zx\,dA$. Then it follows that the moment of the forces acting on the entire base is $\int_A zx\,dA$. Since this must be equal to the moment of the load about the same center, we have

$$Pe = \int_A zx\,dA. \tag{2}$$

From similar triangles, we have

$$(x - s_1) : (s_2 - s_1) = \left(z + \frac{d}{2}\right) : (d).$$

Solving this for x, we find

$$x = \frac{s_1 + s_2}{2} + \frac{z}{d}(s_2 - s_1).$$

Putting this value for x in equation (2), we get

$$Pe = \int_A \left(\frac{s_1 + s_2}{2} z\,dA + \frac{s_2 - s_1}{d} z^2\,dA\right). \tag{3}$$

But, $dA = bdz$; therefore

$$\begin{aligned} Pe &= \int_A \left(\frac{s_1 + s_2}{2} bz\,dz + \frac{s_2 - s_1}{d} bz^2\,dz\right) \\ &= \frac{b}{2}(s_1 + s_2)\int_A z\,dz + \frac{b}{d}(s_2 - s_1)\int_A z^2\,dz. \end{aligned}$$

Now z may have values between $+ d/2$ and $- d/2$. By integrating and imposing these limits, we get

$$Pe = (s_2 - s_1)\frac{bd^2}{12}. \tag{4}$$

Using equations (1) and (4) and solving for s_1 and s_2, we get

$$s_1 = \frac{P}{bd}\left(1 - \frac{6e}{d}\right), \tag{5}$$

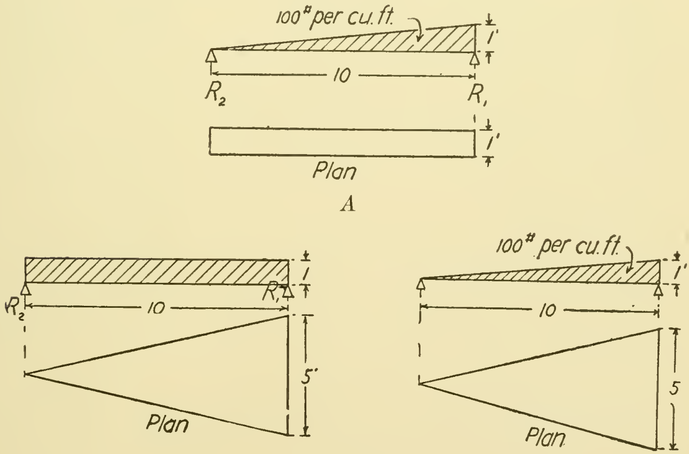
$$s_2 = \frac{P}{bd}\left(1 + \frac{6e}{d}\right). \tag{6}$$

These results will be discussed more fully in Chapters XX and XXII.

106. Summary. In the problems presented in §§ 103–105, the same elements constantly recur. Uniformly varying forces are producing a moment which must be evaluated in order to reach the solution. This moment is expressed as some function of $\int_A z^2\,dA$. (See equation (4), § 103A; equation (5), § 103B; equation (2), § 104.) It will be noted that the laborious part of these solutions arises in the evaluation of this integral. In the next chapter we will develop a means for solving such problems with greater ease, but the essence of all solutions must be much the same as of those in §§ 103–105.

PROBLEMS

1. What are the reactions in Fig. 187*A*?
2. A beam, 12″ × 12″ and 10′ long, is made of material which varies uniformly in heaviness from 0 at one end to 100 lbs. per cu. ft. at the other end. What end reactions will be required to support it?



B Fig. 187 *C*

3. What are the reactions in Fig. 187*B*, (*a*) if the material weighs 100 lbs. per cu. ft., (*b*) if the heaviness of the material varies uniformly from 0 at the left end to 100 lbs. per cu. ft. at the right?
4. What are the reactions in Fig. 187*C*?

5. An elliptical plate whose major axis is 15′ 0″ and whose minor axis is 10′ 0″ is bisected along the minor axis. The semi-ellipse is loaded with a uniformly distributed load of 100 lbs. per sq. ft., and is supported at the ends of the axes. What are the reactions?
6. In problem 5, let the load vary from zero at the straight side to 150 lbs. per sq. ft. at the far support. What are the reactions?
7. In Fig. 185, let the valve plate be an equilateral triangle 2′ 0″ on each side. The hinged side is 6′ 0″ below the water surface and is horizontal. The apex of the triangle is below the hinged side and is held in place by a horizontal force, P. Determine the force P and the pressure on the hinges.
8. In Fig. 185, let the plate be semi-circular, 4′ 0″ in diameter and hinged along the diameter which is at the top of the plate and 3′ 0″ below the surface of the water. A horizontal force P acts on the vertical radius of the plate, 1.75′ below the hinges. How great must P be in order to hold the plate in position?

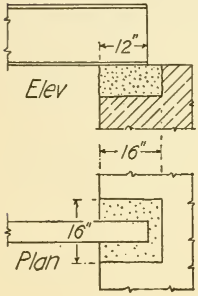
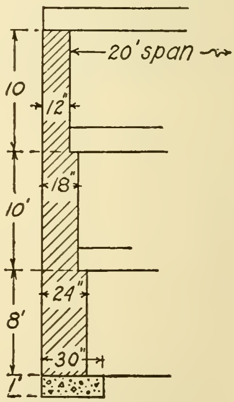


Fig. 188

9. A stone bearing block 2′ 0″ × 2′ 0″ carries a column the load from which is 40 tons. If the column is set 2″ off center but on one axis of the block, what will be the maximum and minimum unit pressures under the block?
10. The end of a 15″ × 42 lb. I beam rests on a stone bearing block as shown in Fig. 188. What are the maximum and minimum unit pressures under the block (*a*) if mortar is figured as carrying tension; (*b*) if mortar is not figured as carrying tension. Reaction of beam 10,000 lbs.
11. A timber 6″ × 6″ is built into a brick wall to a depth of 1′ 6″. It projects from the wall 4′ 0″ and carries at its end a load of 100 lbs. What is the greatest unit pressure between the wall and timber. Neglect weight of timber. In working this problem let it be assumed that the unit pressures vary uniformly from a maximum (upward) pressure at the edge of the wall, to zero at some point within the wall, and then again to a maximum (downward) at the inner end of the timber.



Floors, 100 lbs. per sq. ft.; walls, 150 per cu. ft.

Fig. 189

12. Figure 189 shows a section through the party wall of a city building. What is the maximum and the minimum unit pressure on the soil, in tons per sq. ft.?

CHAPTER XII

MOMENT OF INERTIA

107. Introduction. In Chapter XI the problem of the moment set up by uniformly varying forces was discussed by means of a number of specific cases. It was noticed that in each case the solution, at some stage, involved an expression of the general form $\int_A z^2\,dA$. A more general consideration will show that this expression is bound to arise whenever we attempt to evaluate the moment effect of uniformly varying forces distributed over a given area; that is to say, in a large proportion of the problems having to do with beams, columns, eccentric loading, and various other matters.

Figure 190*A* shows an irregular area acted upon by uniformly varying forces. Let it be required to determine the tendency of these forces to produce rotation about the axis 1–1. (Fig. 190*B*.) Let s_1 and s_2 represent the *intensities* of the loading at the minimum and maximum values. Then x represents the intensity of loading on the elementary area dA (Fig. 190*A*). Hence the *total* load on dA will be $x\,dA$ and the *moment* of this load (about 1–1) will be $xz\,dA$. It will be noticed that *x is a function of z.* The value of x (in terms of z) can be established by reference to Fig. 190*B* in which

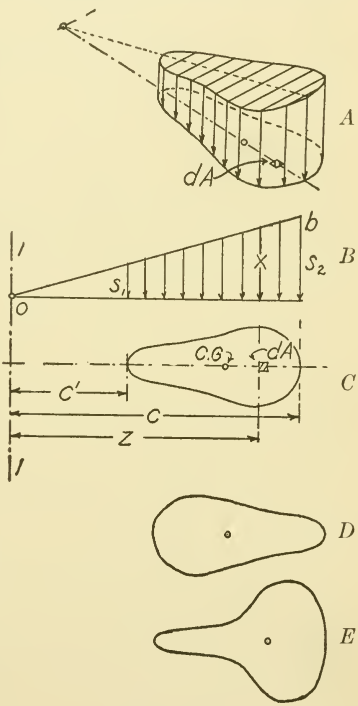


Fig. 190

$$x : z = s_2 : c \qquad \text{and} \qquad x = \frac{s_2}{c} z.$$

Then the moment of the load on the elementary area becomes

$$\frac{s_2}{c} z^2 \, dA$$

and the moment of the entire load is

$$\int_A \frac{s_2}{c} z^2 \, dA.*$$

In this expression s_2/c is a constant and can therefore be put outside the sign of integration. Then we have:

(1) The moment of the load (about 1–1) $= \dfrac{s_2}{c} \displaystyle\int_A z^2 \, dA.$

In this expression s_2/c represents the slope of the line *ob*, i.e., the rate of variation of the loading while $\int_A z^2 \, dA$ represents the effect of the size and shape of the area. If the axis 1–1 were chosen either nearer to or farther from the given area, the values of each of these quantities

$$\frac{s_2}{c} \qquad \text{and} \qquad \int_A z^2 \, dA$$

would be affected thereby but the essential form of the result would not be altered.

Thus there are three distinct quantities which affect the values in equation (1) above:

(1) The rate of variation of the loading.

(2) The position of the axis about which moments are figured.

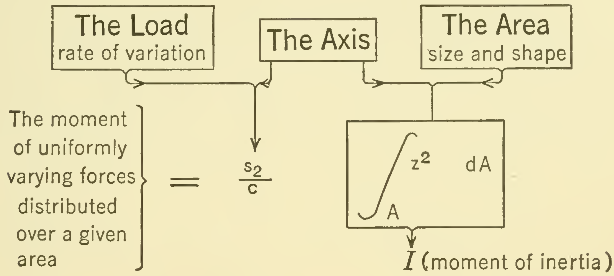
(3) The size and shape of the area.

The diagram given below is intended to express visually the way in which these quantities are related.

From the above discussion and from the summary in § 106 it should be evident: (1) That $\int_A z^2 \, dA$ is a quantity that will inevitably occur in the discussion of such problems. (2) That its

* See footnote, page 63.

amount in any given problem is dependent on the size and shape of the area and the position of the axis of rotation; and on



nothing else. (3) that it is, therefore, a property of an area; just as $\int_A x\,dA$, *the static moment* (§ 53), is a property of an area. (4) That it will be convenient to give it a name and work out some standard cases for reference.

This property of an area ($\int_A z^2\,dA$) is called its ***moment of inertia.*** It has already been pointed out (§§ 52 and 53) that the terms "Center of Gravity" and "Static Moment" when applied to an area cannot be regarded as having the same literal significance as when they are applied to bodies having definite weight. In the same way, *moment of inertia* is more nearly descriptive when applied to a solid than when applied to an area which, from its very nature, can have neither moment nor inertia. But while the term cannot be said to be accurately descriptive, it has, nevertheless, been given a very definite meaning even when applied to an area. In a purely mathematical sense the moment of inertia of an area may be described as the limit of the sum of the products obtained by multiplying each elementary area (composing the given area) by the square of its distance from an axis. This limit has for its value the definite integral as used above.

In a physical sense the moment of inertia of an area may be conceived of as being a factor which indicates the influence of the area itself in determining the total rotating moment of uniformly varying forces applied over the area. The nearest related

quantity is static moment which merely involves the distance from the axis, rather than its square.

108. Units of Measurement. Since the moment of inertia of an area is the sum of products each of which consists of an area and a distance squared, it is evident that it is made up of four linear units. If the linear unit used is the inch, moment of inertia will be expressed as $(\text{inches})^4$. Such quantities are sometimes called "biquadratic inches"; though "inches to the fourth power" is perhaps the more usual term.

No real area can have a negative moment of inertia; for the distance element (which can have negative values) enters as its second power (z^2) and dA itself is positive.

109. Effect of Size and Shape. If the areas in Figs. 190D and 190E are the same *size* as that in Fig. 190C, and if they are similarly loaded, it is evident that the rotating moments in the three cases would vary widely. The proportion of elementary areas with large lever arms, in Fig. 190D, is much less than in Fig. 190C. Hence the moment of inertia of the area is less as is also the moment of the varying forces. The reverse is true in Fig. 190E.

Thus it is easy to foresee that the moment of inertia of an area will be large when the area is large; when it contains a large proportion of elementary areas as far from the axis as possible, or when the axis is chosen far from the center of gravity of the area; and it will be small under the opposite conditions.

110. Methods of Computation. The determination of moment of inertia is essentially an integrating operation (compare with § 54). But once the values are worked out for the typical geometrical figures, the results so obtained can be combined, and the axes transferred by ordinary operations of addition, subtraction, and multiplication. In the five sections following the moments of inertia of a few typical figures are derived and methods are set up for combining them. The properties of various other shapes may be found listed in the standard books of reference. With these materials at his disposal the student should be able to solve all ordinary problems.

Hereafter the symbol I will be used to designate moment of inertia.

111. Moment of Inertia for a Rectangle: Axis through Center. Let it be required to find I for a rectangle of known size and shape, using an axis through its center, as shown in Fig. 191. Choose an elementary strip (shown shaded in the figure) at a distance z from the axis and of width dz. Its area is $b\,dz$ and the I for the entire area will be $\int_A z^2 b\,dz$, between the limits $+d/2$ and $-d/2$ (by definition. See § 107). Putting the constant b outside the integral sign, this becomes:

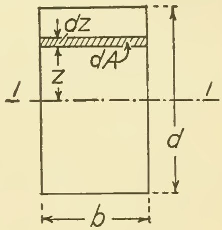


FIG. 191

$$b\int_{-d/2}^{+d/2} z^2\,dz = b\frac{z^3}{3}\Big]_{-d/2}^{+d/2},$$

and when this expression is evaluated it becomes

$$\frac{bd^3}{12}.$$

This expression gives the I for a rectangle in terms of its breadth and length and can be used to simplify the computations in such a case as equation 3, § 105, in which $bd^3/12$ could have been substituted for $\int z^2\,dA$, thereby materially shortening the solution.

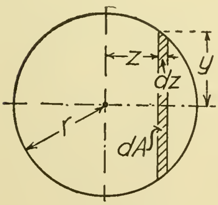


FIG. 192

112. Moment of Inertia of a Circle: Axis through the Center. Refer to Fig. 192. Let the elementary strip of area be chosen as shown shaded in the figure and let its area be expressed by dA. Then by definition (see § 107), the I of the circle is given by the equation

(1)
$$I = \int_A z^2\,dA.$$

The area of the strip chosen is

$$(2) \qquad dA = 2y\,dz$$

and from the properties of a circle $z^2 + y^2 = r^2$, which may be transformed to read $y = (r^2 - z^2)^{1/2}$. Now substituting this value of y in equation (2),

$$dA = 2(r^2 - z^2)^{1/2}dz,$$

and substituting this value of dA in (1),

$$I = \int_{+r}^{-r} 2z^2(r^2 - z^2)^{1/2}dz.$$

Integrating this:

$$I = 2\left[-\frac{z}{4}(r^2 - z^2)^{3/2} + \frac{r^2}{8}\left(z(r^2 - z^2)^{1/2} + r^2\sin^{-1}\frac{z}{r}\right)\right]_{+r}^{-r},$$

$$(3) \quad I = \frac{\pi r^4}{4} = \frac{\pi d^4}{64}.$$

PROBLEMS

1. Find the moment of inertia, referred to the base, of a triangle whose base is b and whose altitude is h.
2. In problem 1, let the axis of reference be parallel to the base and through the center of gravity.
3. Find the moment of inertia of a circle referred to a tangent line.
4. Find the moment of inertia of an ellipse referred to its minor axis. Let the minor axis be a_1 and the major axis a.
5. Find the moment of inertia of a rectangle referred to its shorter side.

113. Transfer of Axes. From the discussions in §§ 107–109, it is evident that moment of inertia is an important factor in comparing the effect of different distributions of area. This is especially true in connection with the design of beam sections. In Fig. 241 is shown a typical plate girder section, composed of plates and angles. In order to determine the strength of the girder it is necessary (as will be shown in Chapter XIII) to determine the moment of inertia of the section. The moment of inertia of each part, referred to an axis through *its* center of gravity, is known (§ 111). These must be transformed so that all are referred to the *same* axis. It thus becomes necessary to provide a means whereby the axis can be freely shifted. In this article we will

deal only with the transfer of axes from an axis *through the center of gravity* of an area to an axis parallel to the first. Transfer of axes through an angle is treated in § 223.

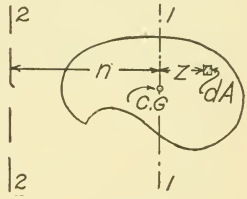


FIG. 193

In Fig. 193, let I be the moment of inertia of the area, referred to the *gravity* axis 1–1, let A be its area and let the moment of inertia (I_0) about the axis *2–2* be required. The elementary area dA is distant $(n + z)$ from the new axis and the I about the new axis is then:

$$I_0 = \int_A (n+z)^2 dA = \int_A n^2 dA + \int_A 2nz\, dA + \int_A z^2 dA$$

$$= n^2 \int_A dA + 2n \int_A z dA + \int_A z^2 dA.$$

But since $\int_A dA$ is the total area and since $\int_A z\, dA$ is 0 when the axis of z is through the center of gravity (§ 52), then the above expression becomes

$$I_0 = An^2 + \int_A z^2 dA = An^2 + I.$$

If the dA had been chosen to the left of the gravity axis, the result would have been the same. Another way of arriving at the same result is to consider the fact that in moving the axis we have added to the average z an amount called n, and this addition has affected a total area of A. Then, since moment of inertia is area times distance squared, the *additional* moment of inertia is An^2 and the total moment of inertia about the new axis is $I + An^2$.

From the above it is plain (since An^2 cannot be negative) that the moment of inertia about a gravity axis is less than that about any parallel axis.

114. Irregular Shapes—By Addition. Moments of inertia of irregular shapes may be found by processes of addition and

subtraction, identical in principle with those used in finding centers of gravity.

Let it be required to find the I for the area in Fig. 194 referred to the axis X–X. Divide the angle into two strips, $6'' \times 1''$ and $4'' \times 1''$. Then the I for the first strip, referred to axis *1–1*, is

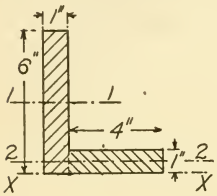


Fig. 194

$$\frac{1 \times 6 \times 6 \times 6}{12} = 18.$$

Transferring this to axis X–X, we get $18 + (6 \times 1)(3)^2 = 72$. Similarly the I for the $1 \times 4''$ strip about axis *2–2* is

$$\frac{4 \times 1 \times 1 \times 1}{12} = \frac{1}{3};$$

and transferring this to axis X–X, we get $\frac{1}{3} + (4 \times 1)(\frac{1}{2})^2 = 1\frac{1}{3}$. Adding these two items we get I for the angle to be $73\frac{1}{3}$ (ins.)^4 referred to the axis X–X.

115. Irregular Shapes—By Subtraction. It was pointed out in § 108 that I cannot be negative. Nevertheless, it is sometimes convenient to consider that a hole has a negative I. In what follows a negative I indicates in reality something not present in the area first considered.

Thus in Fig. 194, let us consider a positive area $6'' \times 5''$ and then subtract an area $5'' \times 4''$. The I for the first area about its gravity axis will be

$$\frac{5 \times 6 \times 6 \times 6}{12} = 90;$$

and transferring to X–X, this becomes $90 + (6 \times 5)(3)^2 = 360$. For the second area, the I about its gravity axis is

$$\frac{4 \times 5 \times 5 \times 5}{12} = 41\tfrac{2}{3};$$

and transferring this to X–X, we get $41\frac{2}{3} + (4 \times 5)(3\frac{1}{2})^2 = 286\frac{2}{3}$. Subtracting the second from the first we get $73\frac{1}{3}$ (ins.)^4, as before.

116. Summary. The following summary may be found useful in clarifying and generalizing the ideas developed in §§ 107–115.

(1) Moment of inertia is a property of an area ($\int_A z^2\, dA$). § 107.

(2) It is expressed in linear units to the fourth power; usually as (inches)^4. § 108.

(3) It cannot have negative values. § 108.

(4) It may be computed with reference to any axis.

(5) It may be transferred from a gravity axis to any parallel axis. § 113.

(6) It is less when referred to a gravity axis than when referred to any parallel axis. § 113.

(7) Moments of inertia may be added and subtracted when (and only when) all are computed with reference to the same axis.

(8) Moments of inertia are large when either the area is large or when it is so disposed that a large proportion of its elements are remote from the axis. § 109.

(9) Further general principles are given in § 228.

PROBLEMS

1. Determine the moment of inertia of a square whose side is d, referred to a diagonal.
2. Determine the moment of inertia of the trapezoid in Fig. 195, referred to EF.
3. Determine the moment of inertia of the trapezoid in Fig. 196, referred to GH.
4. Determine the moment of inertia of a rectangle 12″ × 16″, referred to one of the 12″ sides.
5. Determine the moment of inertia of an equilateral triangle 6″ on a side,

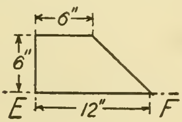


Fig. 195

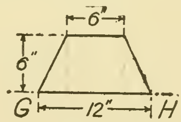


Fig. 196

referred to a line bisecting an apex.

6. Determine the moment of inertia of the area shown in Fig. 126, referred to an axis coinciding with the top edge.
7. Determine the moment of inertia of the area shown in Fig. 128, referred to AB.
8. Determine the moment of inertia of the area in Fig. 129, referred to CD.

9. Determine the moment of inertia of the area, Fig. 130, referred to a horizontal axis through the center of gravity.
10. Determine the moments of inertia of the I section, Fig. 347, referred to each of the axes of symmetry.
11. Determine the moment of inertia of the section, Fig. 197*A*, referred to axis *1–1* (through the center of gravity), and also to axis *2–2*.
12. Determine the moments of inertia of the beam section, Fig. 260, referred to each of the axes of symmetry.
13. Determine the moment of inertia of the beam section, Fig. 240, about a horizontal axis through its center of gravity.

A Fig. 197 *B*

14. Determine the moment of inertia of the plate girder section, Fig. 241, referred to a horizontal axis through the center of gravity.
15. Determine the moments of inertia of the column section, Fig. 197*B*, about each of the axes of symmetry.

$I = I_{cc} + Ad^2$

CHAPTER XIII

BEAMS—TOTAL STRESSES

117. Introduction. The word *beam* is commonly used in two quite different senses. In one sense it is applied to materials worked into a given size and shape. Thus a piece of timber, say 6″ × 8″ × 12′ 0″, is called a wooden beam; or a steel member with an I shaped section and 8′ 0″ long is called a steel beam,—regardless of how the material is used, or whether it is in use at all. In another sense, the word beam is used to indicate any structural member of some considerable length, resting in an approximately horizontal position on its supports and carrying loads which are substantially vertical. In the following chapters we will use the word beam to denote any structural member which is stressed by the action of coplanar forces oppositely directed, and which are not near to one another. (Compare with § 41.) Such forces tend to bend the member on which they act rather than to elongate or shorten it. The concept of a beam then will have to do with its function rather than with its shape or position. Primarily it is a member which resists bending.

Common experience tells us that when a piece of material is bent, its deformed shape is a curve. We may expect to find that the stresses which produce this complex deformation are complex themselves. Nevertheless the solution for such cases will be found to proceed along familiar lines. First of all the relation between loading and reactions is studied by means of the laws of equilibrium (§ 32) and then the stresses resulting from the action of the external forces are determined by means of the free body method (§ 23).

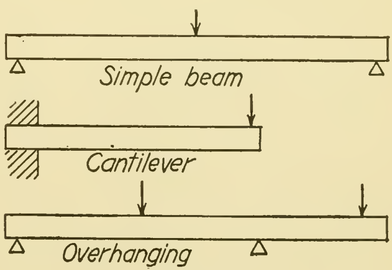


FIG. 198

A classification of beams which is in common use is based on the way in which the beam is supported. When the beam merely rests on two supports (Fig. 198, upper), it is called a *simple beam.* When it projects from a single support, as when built into a wall or pier (Fig. 198, middle), it is called a *cantilever.* If one or both ends of a simple beam project beyond a support (Fig. 198, lower), the beam is an *overhanging beam.* Besides these there are *continuous beams* and *restrained beams,* which are defined and explained in Chapter XVIII.

118. General Ideas.—Forces and Stresses. The laws of equilibrium (§ 32) form the basis of the study of beam stresses. If the beam is in equilibrium, there is a static relation between the loads and reactions (§ 41). By means of this relation the reactions due to a given set of loads ordinarily can be determined.*

Since the correct determination of reactions is the starting point for the entire study of beam stresses, the student should make sure, before proceeding further, that he has a thorough mastery of this subject.

The most general sort of an idea of the stresses in a beam may be found by reference to Figs. 199 and 200. In Fig. 199 the

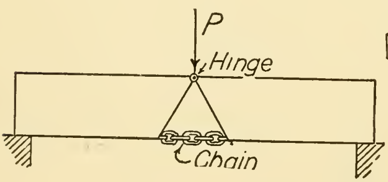


Fig. 199

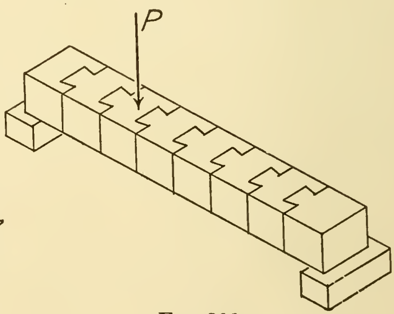


Fig. 200

load P tends to depress the hinge and tighten the chain. It will be recognized that this beam will collapse at once if it is turned upside down. This indicates that the stresses in such a beam are compressive at the top and tensile at the bottom. Any one who

* For the exceptional cases see Chapter XVIII.

has used a saw has verified these statements from his own experience.

Figure 200 shows a beam composed of dove-tailed blocks. Such a member can carry tension or compression; but if called upon to act as a beam, it will fail by a vertical slipping in the dove-tailed joints. If the joints were lightly glued, failure when it occurred would be due to shearing along the glued surfaces.

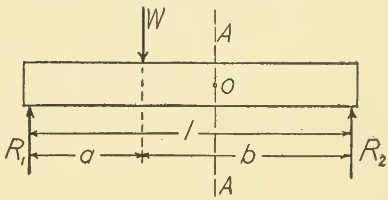


Fig. 201

In all but the rarest cases the tensile, compressive, and shearing stresses noticed above occur on every section cut through a beam. The method of study to be followed involves treating the tensile and compressive stresses together (Chapter XIV) and the shearing stresses quite separately (Chapter XV). Hence it is highly important that from the start the student keep the tendency to rotate (Fig. 199) quite distinct and separate in his mind from the tendency to move vertically (Fig. 200). Confusion on this point is the beginning of most of the difficulties encountered in the study of beam stresses.

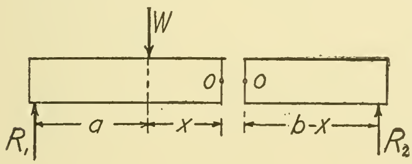
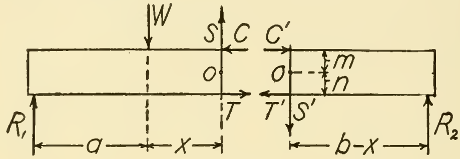


Fig. 202

Figure 201 shows a loaded beam and the resulting reactions. If a part of this beam is taken as a free body (Fig. 202), the external forces are not in equilibrium. Since R_1 is less than W, the summation of the vertical forces is not zero, nor is the summation of the moments about o equal to zero. This shows that there must be stresses acting on the cut section to balance the external forces.

In Fig. 203A, stresses have been indicated on the cut section to balance the external forces on the left-hand portion of the beam. Here $R_1 + S$ may equal W. Also (taking o as a center) the

moment of C and T (counterclockwise) may balance the moment of W and R (clockwise). Again C may be equal to T. In this event it can be seen that it is possible for all three of the conditions of equilibrium to be fulfilled. Similarly in Fig. 203B the stresses S', C', and T' balance the upward and counterclockwise tendencies of the external force R_2.



A FIG. 203 B

The stresses S and S' can be shown to be equal. Also the stresses C and C' and T and T' can be proved equal by simple equations based on the conditions of equilibrium. (Let the student work out these proofs.) Hence S and S' consitute a shear on the cut section, while C and T and C' and T' constitute a moment which resists the tendency to rotate.

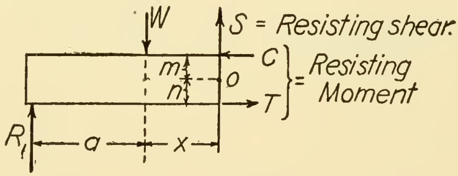


FIG. 204

In Fig. 204, the left portion of the beam is shown with the external forces and internal stresses. The tendency of the external forces to produce vertical motion is called the *vertical shear*. The corresponding stress (S) is the *resisting shear*. The tendency of the external forces to produce rotation is called the *bending moment*, and the internal resistance to rotation (the moment of C and T about o) is called the *resisting moment*.

In the cases treated in §§ 119–124, and in general in the demonstrations and problems which follow, the weight of the beam itself is not considered unless that fact is specially mentioned.

119. Variation in Moment and Shear. It should be noted particularly that the terms *resisting shear* and *resisting moment*, defined above, refer to *total* stresses. For the determination of safe load, for design or for investigation of beams (§ 91), it is necessary that these *total* stresses be transformed into unit stresses, so that their amounts can be compared to the safe unit stresses for the various materials. These problems are worked out in Chapters XIV and XV. Before taking up these problems, it will be necessary to study the relation between various types of loading and the total bending moments and shears, which result from such loads.

Let the beam shown in Fig. 201 be imagined to be cut by a section farther to the right than the section AA. Evidently the bending moment will not be the same as before since the lever arms of the forces producing the moment are changed. Again if the new section is taken to the left of the load, the vertical shear will be different from that shown in Fig. 204. In general a given system of loads and reactions produces shear and bending moment on all sections cut through a beam; but the amounts of each will vary from section to section. The balance of this chapter will be devoted to the study of these variations. In such a study it is usual to draw curves which show the length of the beam along a horizontal line, while the amount of the shear or moment is represented as the ordinate of a curve measured from this line. (See Figs. 215–217.)

In order to make such diagrams with speed, accuracy, and confidence, three things (and three only) are needed: *first,* an accurate knowledge of what shear or moment really is; *second,* a liberal use of the free body method; *third,* a consistent refusal to think of shear when moment is being discussed, and vice versa.*

120. Shear Diagrams. *The vertical shear* (usually referred to as "*the shear*") *is the tendency of one part of a loaded beam to rise or fall with respect to another part.*

This tendency to rise or fall is due to the action on the beam

* As the student proceeds with the study of shear and moment diagrams, it will be wise for him to review this paragraph frequently.

of all the forces, both loads and reactions. Therefore the first step in such a problem is usually to determine the reactions due to a given loading system. In § 118 it was shown that the shear on a given section is the same whether it is computed from a free body taken to the left or to the right of the section. In what follows the free body to the left of the section will usually be considered.

For example, in Fig. 205*A* is shown a beam with loads and reactions. Let a section be cut to the right of the left support (as at *S*) and the left part of the beam be shown as a free body, as

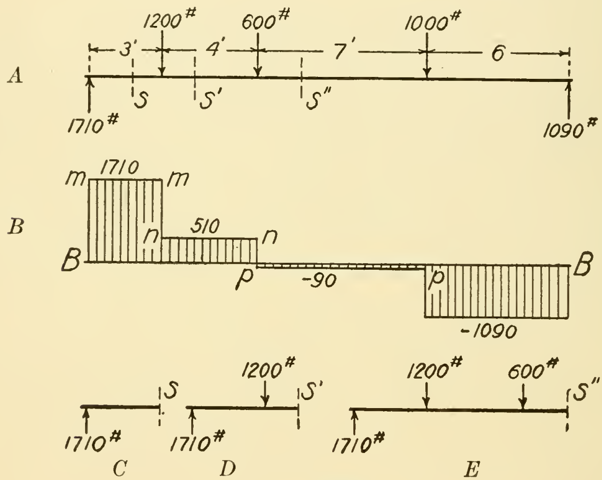


Fig. 205

in *C*. The free body, under the action of the reaction, tends to move upward.* This tendency is measured by the amount of the reaction, 1,710 lbs. Moreover, if our section had been taken anywhere between the reaction and the first load, the same condition would have been found. Thus the shear on any section between these limits is the same and is equal to 1,710 lbs. In *B* the line *BB* is used as a zero or base line. From a point on

* In the beam itself, of course, the reaction has no tendency to actually push the beam bodily upward. But we are here considering the effect of an indicated upward force *on a free body*. Or again, the part of the beam to the left of *S* does actually tend to move upward *with respect to* the part to the right of *S*. See the definition of shear at the beginning of this article.

this line directly below S let a perpendicular be erected which, at some scale, represents 1,710 lbs. This line represents the shear on the section S. We can now draw the line mm to represent the shear on every section between the reaction and the 1,200 lbs. load.

Now let a section be taken at S'. The resulting free body is shown in D. Its tendency to move upward is 1,710 − 1,200 = 510 lbs. This is the amount of shear for any section between the two loads, as shown by nn, in B. A section cut at S'' gives the free body shown in E. This body tends to move downward. Therefore the shear diagram for this section falls below the base line, as shown by pp in B. The rest of the diagram is self-evident.

When the free body at the left of the section tends to move upward, the shear is called positive. We have indicated that condition by an ordinate drawn upward from the base line. Negative shear is then indicated when the free body tends to move downward. This distinction is purely arbitrary.

In Fig. 206 is shown a beam with a uniformly distributed load. A section cut 1′ 0″ from the left support gives the free body shown in C. Its tendency to move upward is $wL/2 - \overset{1(w)}{w}$. A section cut 2′ 0″ from the left would show a shear of $wL/2 - 2w$; and a section cut x feet from the left support would show a shear of $wL/2 - wx$. Evidently the shear changes uniformly and can be represented by a diagram like B. In this diagram the line mm representing the variation in shear is that line whose equation, referred to O as an origin, is

wt/ft

$$(1) \qquad y = \frac{wL}{2} - wx.$$

In Fig. 207 is shown a beam with distributed load which varies from 0 lbs./ft. at the left end to w lbs./ft. at the right. A section cut at a distance x from the left end gives the free body shown in C. The total load on the section is

$$\frac{w}{2L}x^2$$

Triangle ½ bh

$\frac{1}{2} x \left(\frac{wx}{L}\right)$

and the shear is

(2) $$\frac{wL}{6} - \frac{w}{2L}x^2.$$

This equation is represented by the curve in B.

The preceding examples are sufficient to show the principles governing the construction of shear diagrams.

121. Relation Between Loading and Shear. In Fig. 206 it will be noted that the uniform intensity of loading corresponds to a

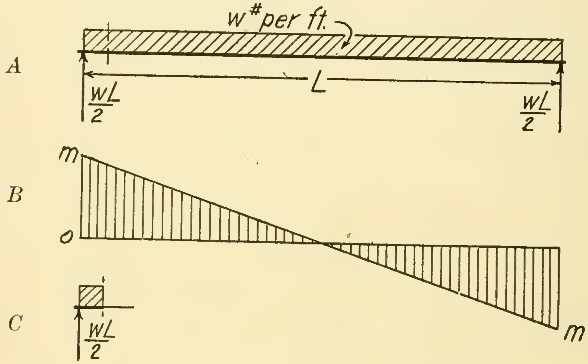


FIG. 206

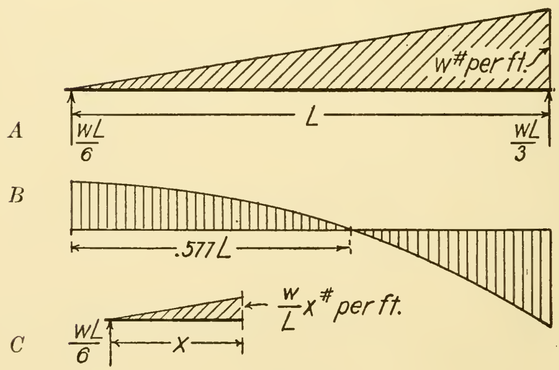


FIG. 207

uniformly sloping shear diagram. Again, in Fig. 207, the variable intensity of loading corresponds to a varying slope on the shear diagram. Where the rate of loading along the beam

is small, the slope of the shear line is small. Where the rate of loading is intense, the shear line is steep.

That this is a necessary relation can be shown by differentiation. For instance, take the two equations for shear in the preceding article:

(1) $$y = \frac{wL}{2} - wx$$

and

(2) $$y = \frac{wL}{6} - \frac{w}{2L}x^2.$$

Differentiating with respect to x, we find

$$\frac{dy}{dx} = w, \qquad \text{and} \qquad \frac{dy}{dx} = \frac{w}{L}x.$$

$S = \frac{wL}{2} - wx$

$M = \frac{wLx}{2} - \frac{wx^2}{2}$

$\frac{dM}{dx} = \frac{wL}{2} - wx$

$\frac{d^2M}{dx^2} = \frac{dS}{dx} = -w$

In each case dy/dx is seen to be equal to the intensity of the loading. In other words *the slope of the shear diagram* is governed by the intensity of the loading. It is sometimes found difficult to understand how the above principle applies to such a case as that in Fig. 205. The application becomes evident at once when

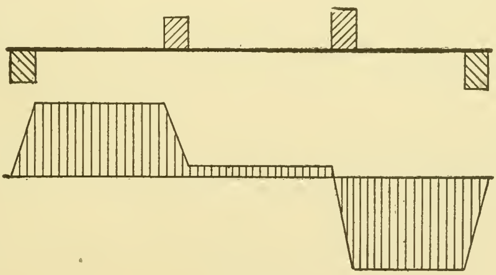


Fig. 208

it is remembered that in no real case are either loads or reactions concentrated at a point but they are distributed over definite distances. Thus in Fig. 208 is shown a beam with loads and reactions thus distributed. The form of the shear diagram and its correspondence to the above principle are both at once evident. However, the form of the shear diagram shown in Fig. 205B is simpler and is the usual form for concentrated loadings.

Let the student show that even in such a diagram the general relations of intensity and slope hold good.

PROBLEMS

NOTE. In each of the following problems show the amount of the shear at each point of sudden change and locate the section of zero shear accurately.

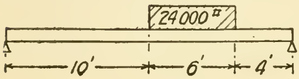


FIG. 209

1. Draw the shear diagram for the beam in Fig. 82, neglecting the weight of the beam.
2. Let the beam in Fig. 82 weigh 50 lbs. per foot. Draw the shear diagram.
3. Let the beam in Fig. 209 weigh 200 lbs. per foot. Draw the shear diagram.
4. Draw the shear diagram for the beam in Fig. 83.
5. Draw the shear diagram for the beam in Problem 11, § 106, using the results obtained in that problem.
6. Write the equations for the lines bounding the shear diagram in the case of a beam whose span is L and which is loaded with a uniformly distributed load of w lbs. per ft. and a concentrated load of P lbs., distant $\frac{L}{4}$ from the left reaction.
7. Write equations like those in Problem 6 for a beam uniformly loaded on the left half of the span only.
8. Write equations like those in Problem 6 for a beam 20′ 0″ long which carries a distributed load of 100 lbs. per foot on the 6′ nearest the left support and a load of 50 lbs. per foot on the balance of the beam.

122. Bending Moment Diagrams. Bending moment is *the tendency of one part of a loaded beam to rotate about the section which separates that part from the rest of the beam.* It is the resultant moment given by all the loads and reactions which occur *on the part of the beam which is considered as free.* The forces are the same as those which produce shear. But now instead of looking for a tendency toward vertical displacement, we are looking for a tendency toward rotation. (See Fig. 210A.)

In Fig. 210C is a free body cut by the section A which is 1′ 0″ from the left reaction. This free body tends to rotate *about the cut section,* in a clockwise direction, under the action of the reaction. The moment (taken about the cut section) is 1,710 lbs. ft. If the section had been cut two feet from the reaction, the

moment would have been 1,710 $\times$ 2 = 3,420 lbs. ft., etc.* Now turn to Fig. 210D. Here the free body (cut by the section A') is

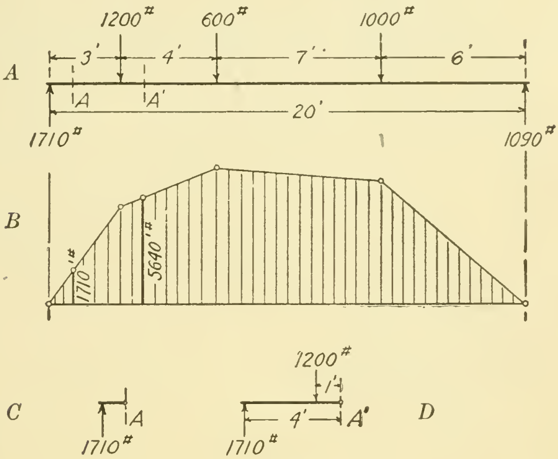


Fig. 210.

acted upon by the reaction and one load. The rotating moment is (1,710 $\times$ 4) – (1,200 $\times$ 1) = 5,640 lbs. ft. Let the student check out the other values on the bending moment diagram. Bending moment is considered to be positive when it tends to produce compression in the top of the beam; in other words, when the curve taken on by the bent beam is concave upwards. (See Fig. 211.)

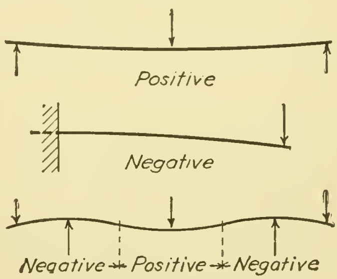


Fig. 211

In Fig. 212 is shown a beam carrying a uniformly distributed load. In C a free body cut by the section A at x ft. from the left end is seen to be under the up-

* In studying bending moment students are frequently confused by the fact that when a free body is cut close to a reaction, often no part of the load appears on it, and hence the load does not appear in the computations for bending moment; nevertheless it is obviously the controlling factor in the situation. It should be noted that the amount and position of the load control the reaction and, through the reaction, they control the bending moment.

ward action of the reaction and the downward action of that part of the load between the section and the reaction. The moment

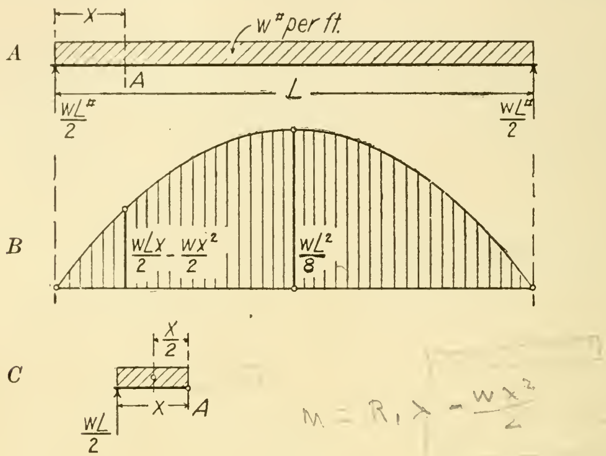


Fig. 212

of the reaction (taken about the cut section) is $(wL/2)x$, clockwise, and that of the load is $wx(x/2)$, counterclockwise. The resultant moment M is given by the equation

$$M = \frac{wL}{2}x - \frac{wx^2}{2}.$$

Considerations of symmetry alone show that this equation must have its maximum value when $x = L/2$, and that for that value of x the bending moment is $wL^2/8$. The same results could have been attained by the usual method of the differential calculus.

In Fig. 213 the free body in C is under the upward action of the reaction which produces a clockwise moment of $(wL/6)x$. The counterclockwise moment is due to the load $wx^2/(2L)$, whose center of gravity is distant $x/3$ from the cut section. This moment is

$$\left(\frac{wx^2}{2L}\right)\frac{x}{3} = \frac{wx^3}{6L}.$$

The resultant moment about the section is

(1) $$M = \frac{wL}{6}x - \frac{w}{6L}x^3.$$

This is the equation of the bending moment curve shown in B. The point of maximum moment can be located by differentiating

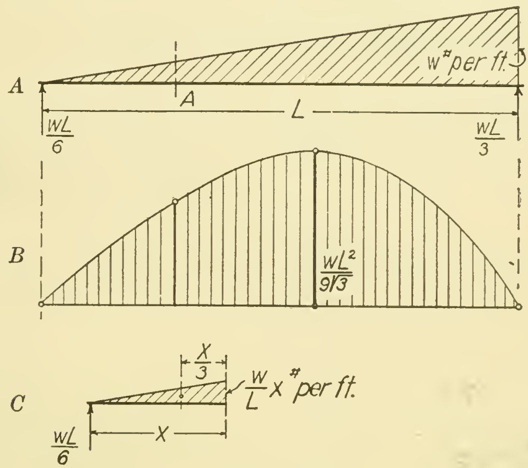


FIG. 213

M with respect to x; this gives

(2) $$\frac{dM}{dx} = \frac{wL}{6} - \frac{3w}{6L}x^2.$$

By placing the right-hand side of this equation equal to zero and solving, we find

$$x = \frac{L\sqrt{3}}{3} = 0.577L.$$

Inserting this value of x in equation (1), we find that the value of the maximum moment is

$$M\ (\max) = \frac{wL^2}{9\sqrt{3}} = \frac{wL^2}{16}\ \text{(about)}.$$

PROBLEMS

NOTE. In each of the following problems show the amount of the bending moment at each point of sudden change, also the position and amount of each maximum value.

1–5. Draw the bending moment diagrams for each of the cases cited in Problems 1 to 5, following § 121.

6–8. Write the equations for bending moment corresponding to the shear equations in Problems 6 to 8, following § 121.

123. Relation Between Shear and Bending Moment. The relation that exists between the loading and the shear was developed in § 121. We shall now show that there is a similar relation between shear and bending moment.

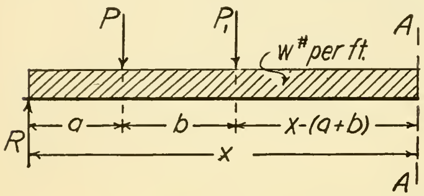


FIG. 214

Let the diagram, Fig. 214, represent a free body cut from the left end of a beam, loaded in *any* manner. The right-hand part of the beam is not needed and is not shown. The vertical shear at the section AA is

$$V = R - wx - P - P_1. \tag{1}$$

The bending moment at the same section is

$$M = Rx - wx\frac{x}{2} - P(x - a) - P_1(x - a - b)$$

$$= Rx - \frac{wx^2}{2} - Px + Pa - P_1x + P_1a + P_1b. \tag{2}$$

Differentiating (2) with respect to x, we find

$$\frac{dM}{dx} = R - wx - P - P_1. \tag{3}$$

This expression is the *slope* of the bending moment curve; i.e., it is the *rate of change* of the bending moment at the section AA. It will be noted that the right-hand sides of equations (1) and (3) are identical. And these equations would be identical if the

section AA were chosen so as to include either more or less loads between it and the reaction. We may conclude therefore that *the shear at any section through a beam measures the rate of change of*

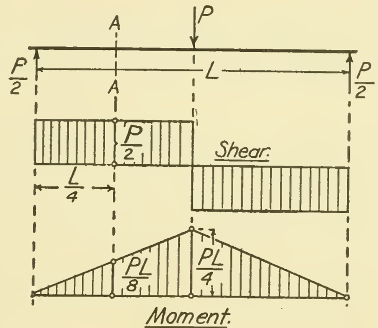


FIG. 215

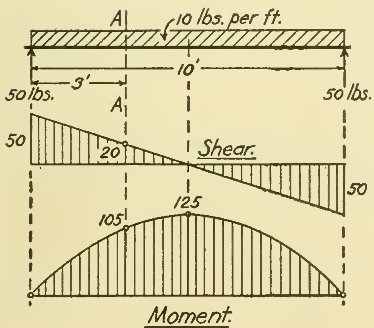


FIG. 216

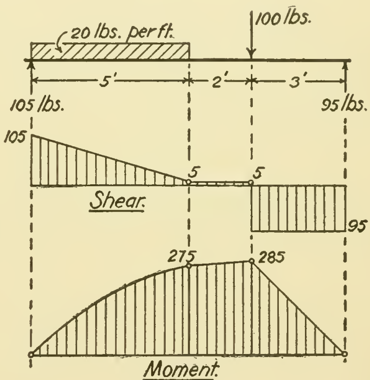


FIG. 217

*the bending moment at that section.** It follows directly from the above, that the bending moment has its maximum value at the section where the shear is zero. Let the student check this statement visually by comparing the diagrams of Figs. 215–217.

124. Area of Shear Diagram. In § 123, it was shown that the shear at any section of a beam is the derivative of the bending moment, with respect to the length (x). In other words,

* This idea is developed from a different standpoint in § 140.

$V = dM/dx$, where V represents the shear and M the bending moment at any section of a beam, distant x from the end.

This relation can be written in the form $Vdx = dM$ and from this it follows that

$$\int_{x_1}^{x_2} Vdx = \int_{x_1}^{x_2} dM.$$

If now it is recalled that $\int Vdx$ can be interpreted as the area under the shear curve and $\int dM$ as the total change in bending moment between any two points, it will be seen that *the area of the shear diagram between two points is the difference between the moments at these points.** Thus, in Fig. 215, the area of the shear diagram between the section AA and the center of the beam is $P/2$ lbs. $\times L/4$ ft. $= PL/8$ lbs. ft. The difference between the bending moments at these two sections is the same amount, $PL/8$. Let the student apply this principle to determine the maximum moment in Fig. 215. Figures 216 and 217 show two other cases of shear and bending moment diagrams prepared to illustrate the above principle.

125. Shear and Moment in Cantilever Beams. In general, the determination of shears and moments in cantilever beams is carried on in the same manner as in the cases already discussed. In the matter of the reactions, however, some further explanation may be worth while.

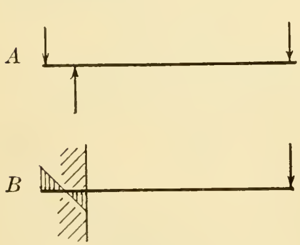


Fig. 218

The reactions of a cantilever beam may occur as localized forces, as shown in Fig. 218A. In that case the reactions may be determined and the shear and bending moment diagrams may be drawn precisely as for simple beams. But when the reactions occur in the form of distributed forces, as shown in Fig. 218B, the case is somewhat

* Provided we interpret $\int ydx$ as the algebraic sum of the areas between a curve and the x axis, where by algebraic sum we mean that those areas above the axis are positive and those below are negative.

different. In the first place, the law governing the distribution of the forces composing the reactions must be known or assumed. When this has been done, the solution can proceed as outlined below.

In Fig. 219 is shown a cantilever beam which is built into its support to a depth of 1′ 8″. The load tends to rotate the beam

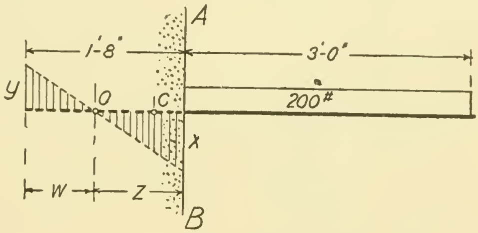


Fig. 219

downward about some point, as o. Let it be *assumed* that the unit pressures composing the reaction vary uniformly from zero at o to a maximum at either side, as shown. Let x and y be the unknown pressure per linear inch. We can then construct four equations to determine w, x, y, and z, as follows:

The *average upward* reaction is $(x - y)/2$ pounds per inch. The total upward reaction is then $20(x - y)/2$. Therefore, from the relation $\Sigma V = 0$, we find

(1) $$10(x - y) = 200.$$

Taking moments about a center at c, which is in line with the center of gravity of the upward reaction, we find from the relation $\Sigma M = 0$,

(2) $$\frac{yw}{2}\left(\frac{2}{3}w + \frac{2}{3}z\right) = 200\left(18 + \frac{z}{3}\right).$$

From the geometric relations involved, we have

(3) $$w + z = 20'',$$

and

(4) $$\frac{x}{y} = \frac{z}{w}.$$

Solving these equations we get $x = 94$ lbs. per inch, $y = 74$ lbs. per inch, $z = 11.2''$, and $w = 8.8''$. The complete shear and bending moment diagrams are as shown in Fig. 220. The

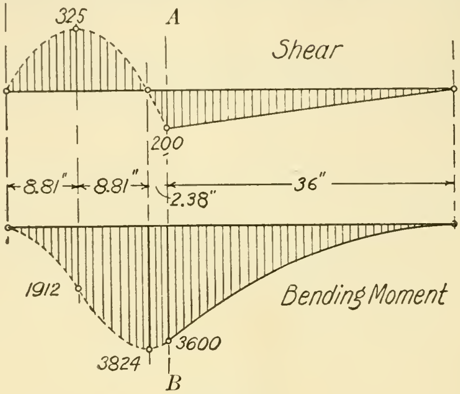


Fig. 220

dotted lines to the left of the vertical line AB (Figs. 219 and 220) show the loading *as assumed* on this part of the beam, and the shears and moments which result from that assumed distribution of the reactions. As a matter of fact, the actual distribution of the reactions in any given case will depend on the elastic properties of the materials and it practically is indeterminate.

In drawing shear and moment diagrams for cantilever beams, it is usual to omit the part at the supported end, as shown in Case 3 of Table III in the Appendix.

PROBLEMS

Note. Draw the shear and bending moment diagrams for the following cases. In each case determine the amount of the shear and of the bending moment at each point of sudden change as well as the position of the zero shear and the amount of the maximum bending moment.

(1) Fig. 73. (2) Fig. 221. (3) Fig. 222.
(4) Fig. 223. (5) Fig. 224. (6) Fig. 225.
(7) Fig. 226(a). (8) Fig. 226(b). (9) Fig. 226(c).
(10) Fig. 227. (11) Fig. 228. (12) Problem 11, § 106.

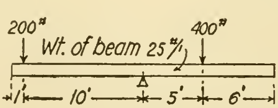


Fig. 221

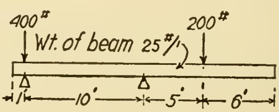


Fig. 222

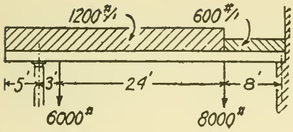


FIG. 223

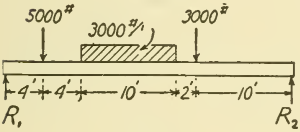


FIG. 224

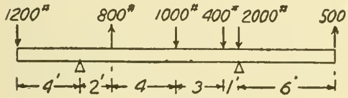


FIG. 225

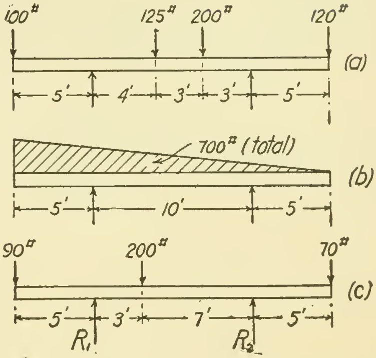


FIG. 226

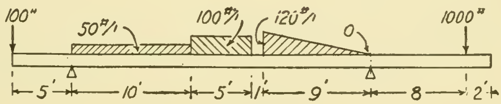


FIG. 227

126. Shears and Moments by Addition. In § 42, it was shown that reactions can be determined by the addition of the component reactions due to each load separately. The same thing can be done with bending moments and shears. In Fig. 229 is shown a case in point. The shear diagram for the load P only is shown by the dot and dash line. That due to the distributed load only is shown by the dotted line, and the final diagram, which is composed from those two, is shown by the solid line. The bending moment diagrams for the individual loads, and that for the combined loads, are constructed similarly. A more complex case is shown in Fig. 270.

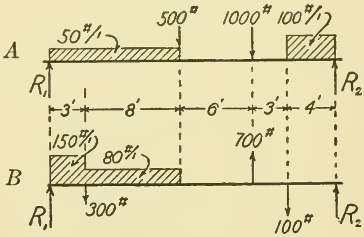


FIG. 228

127. Beam with Uniform Moment. In all cases of bending so far treated, the loads and reactions have been vertical. One

special case remains to be noticed. If a beam is acted upon by longitudinal forces, oppositely directed but not axial, as in Fig. 230A, bending will result. If a free body (Fig. 230B) is cut

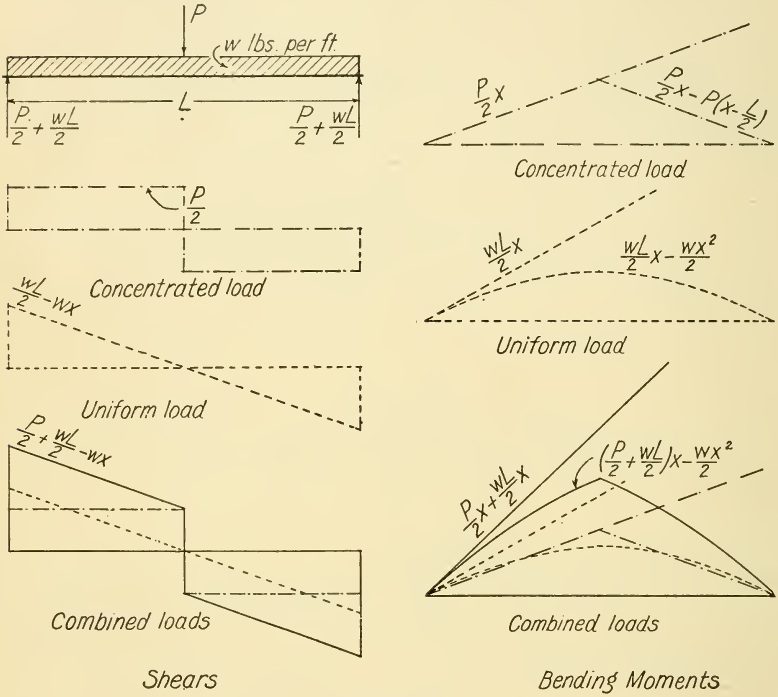


Fig. 229

by any section through the beam, as AA, the tendency to turn about that section is given by Pa. There is no vertical shear on the section. It is thus evident that in such a case we have a zero shear and a uniform moment throughout the beam. (Compare with § 123.)

128. Various Cases of Loading. The student should be familiar with the relations existing between loading, shear, and bending moment, as given in §§ 121 and 123. This familiarity should include not only a recognition of the correctness of the demonstrations, but also a *visual* recognition of the necessary relations between the corresponding curves.

its curve

Table III in the Appendix gives the curves for a number of typical cases of loading. Let the student go over a number of these cases carefully, making sure that he understands how the

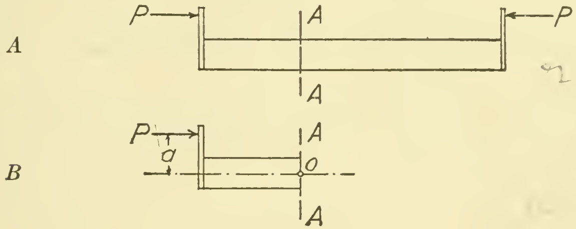


FIG. 230

shape of one curve controls the one below it and is controlled by the one above it, and how the ordinates and slopes and the zero and maximum points relate to one another. As the result of such a study, he should be able, given a loading, shear, or moment curve, to produce the other curves in that series.

PROBLEMS

NOTE. In each of the following cases a loading, shear, or bending moment diagram is given. Draw the other two diagrams which correspond to the one given.

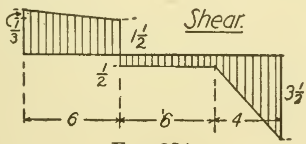


FIG. 231

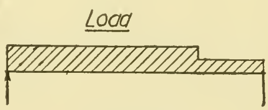


FIG. 232

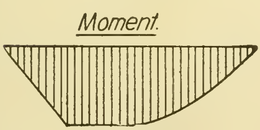


FIG. 233

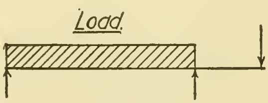


FIG. 234

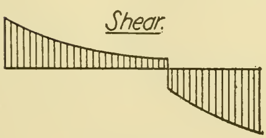


FIG. 235

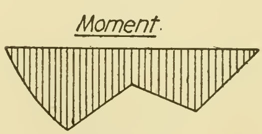


FIG. 236

1. Fig. 231. 2. Fig. 232. 3. Fig. 233.
4. Fig. 234. 5. Fig. 235. 6. Fig. 236.

CHAPTER XIV

BEAMS—UNIT STRESSES IN BENDING

129. Introduction. In the preceding chapter, the effects of beam loading in producing bending moment and shear were studied. Means were found for measuring each effect, and the relations existing between loading and shear, and between shear and bending moment, were established. It was also shown (§ 118) that the bending moment and shear produced by loading are resisted by stresses (the resisting shear and the resisting moment) within the beam and that these stresses, in *total amount*, are equal to the external forces. These statements presuppose that the beam is capable of carrying the imposed load.

Consider a beam under a loading which is slowly being increased. The bending moment increases with the load and calls for an increasing resisting moment to be built up within the beam. This resisting moment is composed of stresses of tension and compression (§ 118) and the maximum that can be developed within a given beam will be reached when the *unit stress* in tension or compression exceeds the *unit strength* of the material. It is proposed, in this chapter, to develop the relations between the *total stresses* and the *unit stresses*, which result. We will then have a means for investigating, designing, or determining the safe load in beam problems.*

Let Fig. 237*A* represent an unloaded beam. When the load P is applied the beam is bent as in Fig. 237*B*, the top face becomes shorter (due to compressive stresses within the beam, as shown in § 118), and the bottom face becomes longer (due to tensile stresses). Moreover if we consider any thin horizontal slice (shown shaded), it is evident that it cannot be either very much shorter or longer than the one above or below it without pro-

*It should be remembered (§ 118) that we intend to deal with the stresses due to bending only. The stresses due to shear will be separately analyzed in Chapter XV.

ducing a horizontal crack in the bent beam. We may conclude, therefore, that the stresses on any given cross section, $A'A'$, vary gradually from compression to tension and that the law of this variation must be understood before definite values for the greatest *unit* stress present in a given case can be determined.

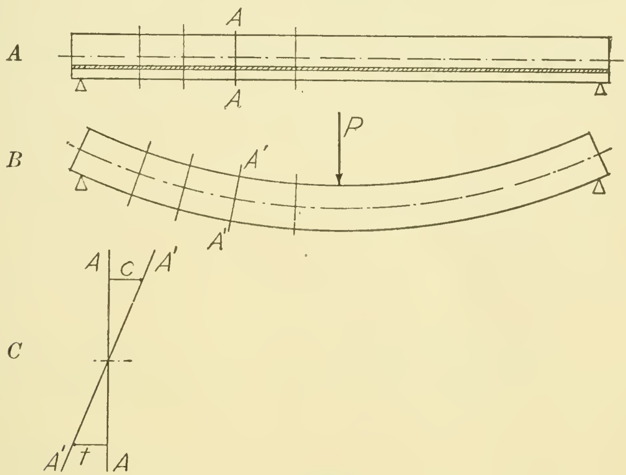


FIG. 237

130. Maintenance of Plane Sections.—Variation in Unit Stress. The theory of beam stresses is a rational one but it is dependent on experiment in two particulars, as stated below.

A. MAINTENANCE OF PLANE SECTIONS. Repeated observations have shown that *when loads are not excessive,* any plane cross section passed through an unloaded beam remains a plane after bending has taken place. Thus in Fig. 237A, the plane section AA on the unloaded beam becomes the plane section $A'A'$ on the bent beam. In Fig. 237C, the relation between the two planes is shown at larger scale. Here the distances c and t represent the deformation of the top and bottom fibers of the beam, due to bending. The deformation of any other fiber is represented by some distance between the planes, measured parallel to c and t. This in effect is the same as to say that the change in the deformation of adjoining fibers *and consequently the change in unit stress* is gradual and uniform.

B. Relation of Moduli of Elasticity. The ordinary materials of construction have the same modulus of elasticity in tension and in compression (see § 70 and Table I). From this it follows that equal deformations indicate equal unit stresses whether in tension or compression.

C. Variation in Unit Stress. Since deformation is proportional to the unit stress which produces it (§ 65), it follows from A and B above, that (1) the *unit* stresses set up in bending vary *uniformly* from a maximum compression on one face of the beam through zero to a maximum tension on the other face, and (2) that the unit stress on any fiber of the beam is proportional to the distance of that fiber from the surface on which the stress is zero.

It should be noted particularly that this law of variation is based on the proportionality between stress and deformation, and that therefore it will hold good only in cases where the unit stresses set up do not exceed the elastic limit of the material (§ 66).

131. Symmetric and Unsymmetric Bending. When the forces producing bending lie in a plane which is also a plane of symmetry of the beam itself (as in Fig. 238), the bending is said to be symmetric.* When the external forces are in some plane which is not a plane of symmetry for the beam itself, the bending is unsymmetric (Fig. 397).

The ordinary theory of bending, which will be developed in this chapter and in Chapters XV to XVIII, treats only of the case of symmetric bending. The general theory of bending, which is treated in Chapter XXII, includes any type of loading, with symmetric bending as a special case.

132. Limits of the Theory. The *general* theory of bending is not applicable to cases in which the unit stresses exceed the elastic limit of the material nor when the material used has a different modulus of elasticity in tension than in compression (§ 130).

The *ordinary* theory of bending is *further* limited to cases of symmetric bending (§ 131).

* A slightly different definition of symmetric bending is given in § 229.

Fortunately these limitations are not serious. No designing of structures is done for unit stresses that exceed the elastic limit of the material, and the usual materials actually do have the same modulus of elasticity in tension as in compression. On the other hand, it is important to keep these limitations in mind when one comes to consider the phenomena of rupture (§ 137).

Again, most of the cases that arise in practice are cases of symmetric bending. Therefore the ordinary theory which follows is usually sufficient. Unsymmetric bending is treated in Chapter XXII.

133. Recapitulation for the Ordinary Theory of Bending. Before starting to develop the equations for unit stresses it will be convenient to review the results thus far obtained, and to arrange them for future reference.

(1) The resisting shear = The vertical shear (§ 118).

(2) The resisting moment = The bending moment (§ 118).

(3) The sum of the tensile stresses = The sum of compressive stresses (§ 118).

(4) The unit stress on a cross section varies uniformly from a maximum in compression to a maximum in tension (§ 130).

(5) The ordinary theory is limited in application as indicated in § 132.

(6) In order to make possible the investigation or design of beams, equations are needed which express numerically the relations between the essential factors in the problem. These factors are: (*a*) The amount and distribution of the load and the span, which determine the shear and bending moment, and (*b*) the material and the size and shape of the cross section of the beam (see page 185), which determine the resistance of a given beam to shear or bending.

We shall now proceed to develop the necessary equations, bending and shear being treated quite separately.

134. Unit Stress.—Bending. Let Fig. 238 represent a side view and cross section of one end of a loaded beam, taken as a free body in the same manner as in Fig. 203. Let the cross section

be considered as representing *any* symmetric shape. In the discussion which follows, no values are to be used which depend on any special size or shape of cross section. Let the surface of no stress, called the ***neutral surface***, be represented by AA, though its position is not yet determined. Let the tensile and compressive *unit* stresses on the section be represented by the

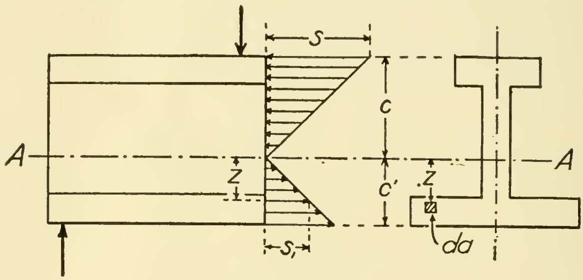


FIG. 238

arrows shown. According to the principles stated above, these arrows are longest when farthest from the neutral surface and vary proportionally to their distances from that surface. Let s represent the largest of these *unit* stresses, let dA be an elementary area or fiber of the cross section distant z from the neutral surface, and let s_1 represent the unit stress on that fiber. From similar triangles, we have

(1) $$s : s_1 = c : z, \qquad \text{or} \qquad sz = s_1 c.$$

Hence the *unit* stress s_1 on a fiber distant z from the neutral surface is

$$s_1 = s\frac{z}{c},$$

and the *total* stress on a fiber distant z from the neutral surface is

(2) $$\frac{s}{c} z\, dA.$$

The total compressive stress on the section, above AA, is

(3) $$\frac{s}{c}\int_c z\, dA,$$

and the total tensile stress on the section, below AA, is

$$\frac{s}{c}\int_{c'} z\,dA. \tag{4}$$

Substituting the values found in (3) and (4) in equation (3) of § 133, we find

$$\frac{s}{c}\int_{c} z\,dA = \frac{s}{c}\int_{c'} z\,dA.$$

This means that $\int z\,dA$ for the area above the axis AA is equal numerically to $\int z\,dA$ for the area below the same axis. But since z is negative for all values below the axis, it follows that $\int z\,dA$ for the entire section must be zero. Now from § 52 we know that $\int_A z\,dA = 0$ if and only if the axis from which z is measured passes through the center of gravity. Therefore, we can conclude that the *neutral surface* of a simple beam passes through the *center of gravity* of the section, provided the case falls within the limitations cited in § 132.

We have now definitely fixed the position of the neutral surface, and we are in a position to attempt an evaluation of the unit stresses. In order to do so let us recall equation (2) of § 133 and attempt to set up a value for the resisting moment of a beam in terms of s. From (2) just above, the total stress on fiber distant z from neutral surface is

$$\frac{s}{c} z\,dA.$$

It follows that the moment of the stress on the same fiber referred to the neutral surface is

$$z\frac{s}{c} z\,dA,$$

and that the *total* moment of *all* stresses on the cross section, referred to the neutral surface, is

$$\int_A \frac{s}{c} z^2\,dA.$$

In this last expression, observe that (*a*) s/c is a constant and may be placed outside the sign of integration, and (*b*) the total expression merely evaluates the resisting moment of the beam (§§ 118 and 129). We may then re-write the expression in the form

$$\text{Resisting Moment} = \frac{s}{c}\int_A z^2\,dA,$$

or, by the definition of moment of inertia, § 107,

$$\text{Resisting Moment} = \frac{s}{c}I. \tag{5}$$

But from (2), § 133,

$$\text{Resisting Moment} = \text{Bending Moment}.$$

Hence

$$\text{Bending Moment} = \frac{s}{c}I. \tag{6}$$

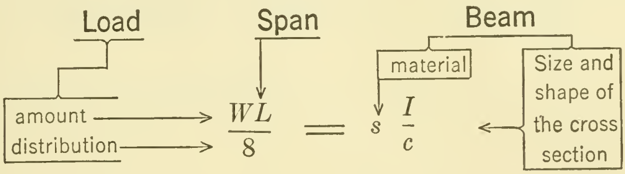
In the above equations s is the unit stress in the *outermost* fiber of the beam which is distant c from the neutral surface. This is the maximum unit stress which occurs anywhere on the section, and hence is usually the controlling factor in design or investigation. However, it is sometimes desirable to determine the unit stress on some other part of the cross section such as dA, Fig. 238. In that case, the relations in equation (1), § 134, may be used. This gives $sz = s_1c$ or $s = s_1(c/z)$. Now substituting this value of s in equation (6) above, we get,

$$\text{Bending Moment} = \frac{s_1}{z}I. \tag{7}$$

This is a general equation for determining the bending unit stress in *any* fiber of a loaded beam.

135. Significance of the Equations. Equation (6), § 134, expresses the relation between the loading on a beam and the unit stress caused by that load. The elements entering into this relation are (1) the load and (2) the beam, with the span as an element common to the load and the beam. If we take the

definite case of a load uniformly distributed over the beam, the bending moment becomes $WL/8$ (§ 122) and equation (6) becomes $WL/8 = s(I/c)$. The diagram given below is intended to bring out the relations between these factors more clearly.



All of the relations are evident except possibly that between s and the material. Since s expresses the unit stress on the outermost fiber, the allowable unit stress on a given material will limit the possible value for the resisting moment or, if the material is not given, the value determined for s may limit the choice of material. Thus if we wish to know how great a resistance to bending can be offered by a given beam, we can use equation (5), § 134, replacing the right-hand side of the equation with values appropriate to the given beam. On the other hand, if we wish to know how great a stress will be produced in a beam of a given size by a given load, equation (6) will be used, treating s as the unknown quantity.

It is not unusual to find the ideas expressed in equations (5) and (6), § 134, consolidated into one expression:

$$M = s\frac{I}{c}, \tag{1}$$

in which M is used indifferently to represent either bending moment or resisting moment as occasion may require.

In this last equation M is expressed in *pound inches;* s is in pounds per square inch; I is in inches to the fourth power, and c is in inches. Hence each side of the above equation is in pound inches. The quantity I/c expresses the value of the size and shape of the cross section (see diagram above) and hence is called the ***Section Modulus.***

136. Applications of the Equation. A. Design of Wooden Beam. Let it be required to design a wooden beam to span 16′ 0″ and carry a uniformly distributed load of 12,000 lbs.

The bending moment of the load is

$$BM = \frac{12{,}000 \times (16 \times 12)}{8} = 288{,}000 \text{ lb. ins.}$$

The beam to be used must have a resisting moment equal to the bending moment when the maximum unit stress in the beam is not greater than the allowable unit stress, say 1,000 lbs. per sq. in. Then

$$RM = 1{,}000\frac{I}{c}.$$

Then from equation (2), § 133

$$1{,}000\frac{I}{c} = 288{,}000,$$

$$\frac{I}{c} = 288.$$

It is then necessary to provide a section such that I/c will equal 288. Now the section of a wooden beam is usually rectangular and the I for a rectangle equals $bd^3/12$ and c equals $d/2$. Therefore I/c equals $bd^2/6$, which, in this case, should be equal to 288. If now b is chosen as 12″, d will work out 12″; or any other combination of b and d will be satisfactory so long as $bd^2/6 = 288$.

B. Safe Load on a Steel Beam. Let it be required to determine the safe distributed load on a 12″ × 40 lb. I beam on a span of 20′ 0″. The safe fiber stresses on steel is 16,000 lbs./sq. in.; and, by reference to the handbook, the section modulus (I/c) of the given beam is found to be 44.8 inches cubed. The resisting moment of the beam is then $44.8 \times 16{,}000 = 716{,}800$ lb. ins. Now the BM of the load may not exceed this amount and still keep s at 16,000 lbs./sq. in. But the bending moment of the distributed load is

$$\frac{WL}{8} = \frac{W \times 20 \times 12}{8} = 30W \text{ lb. ins.,}$$

which, in this case, should not exceed 716,800 lb. ins.; then $30W = 716{,}800$ and $W = 23{,}890$ lbs., which is the required safe load.

C. Investigation. Again suppose that a circular steel beam 4″ in diameter and 10′ 0″ long carries a load of 2,000 lbs. concentrated at the center of the span. Is it safe? Here

$$BM = \frac{2{,}000 \times 10 \times 12}{4} = 60{,}000 \text{ lb. ins.}$$

$$*I = \pi \frac{d^4}{64} = 12.56, \qquad \text{and} \qquad \frac{I}{c} = 6.28.$$

Then $60{,}000 = 6.28s$ and $s = 9{,}550$ lbs./sq. in. This, being less than 16,000 lbs. per sq. in., is a safe unit stress for steel.

D. Investigation. — Irregular Section. Let Fig. 239 represent the cross section of a cast iron beam which is used to carry a uniformly distributed load of 20,000 lbs. on a span of 10′ 0″. The safety of the beam is to be investigated.

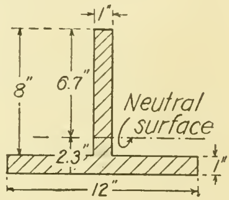


Fig. 239

From § 52 the center of gravity of the section is found to be as shown in the drawing. Then from § 114, the I for the section about an axis through the center of gravity is computed. It is found to be 141 (in.)^4. The bending moment of the load is

$$\frac{20{,}000 \times 10 \times 12}{8} = 300{,}000 \text{ lb. ins.}$$

The maximum unit stress on the outermost fiber *at the top* of the section can now be found from the equation

$$300{,}000 = s\,\frac{141}{6.7},$$

$$s = 14{,}250 \text{ lbs. per sq. in., compression.}$$

The maximum unit stress on the outermost fiber *at the bottom* of

* See § 112 or a table of moments of inertia.

the section is given by

$$300{,}000 = s' \frac{141}{2.3},$$

$$s' = 4{,}900 \text{ lbs. per sq. in., tension.}$$

Each of these unit stresses is less than the allowable for the material (Table I). Therefore the beam can be considered as safe. However, it may be noted that the actual unit stress in tension is about 98 per cent. of the allowable stress, while that in compression is 89 per cent. of the allowable stress. Thus the beam is stronger on the compression side than on the tensile side. By a proper readjustment of the section, the strength could be made the same for both sides.

From equation (7), § 134, it is evident that the unit stress at *any* point on the cross section is proportional to the distance of that point from the neutral surface. Thus in the above beam the unit stress at the joining of the flange and web is found from the relation

$$300{,}000 = s'' \frac{141}{1.3}, \qquad s'' = 2{,}770 \text{ lbs. per sq. in.,}$$

or, by proportion, from the equation $(4{,}900/2.3) \times 1.3 = 2{,}770$.

PROBLEMS

Note. In all of the problems dealing with beams, let the weights of the beams be neglected, except when otherwise specified. When details concerning steel shapes are needed, consult a handbook.

√ 1. A wooden beam 2″ broad and 12″ deep spans 15′ 0″ and carries a uniformly distributed load of 2,560 lbs. What is the maximum unit stress in the beam?

√ 2. In Problem 1, what is the unit stress at a point 4′ 0″ from the left end and 3″ above the neutral surface?

3. What load could the beam in Problem 1 safely carry if turned so that the 2″ dimension is the depth of the beam?

√ 4. Design a wooden beam to carry a central load of 3,000 lbs. on a span of 8′ 0″.

5. A simple beam of concrete is 12″ × 12″ in cross section and carries its own weight only. What is its length if the maximum tensile unit stress due to bending is just equal to the safe unit stress?

√ 6. What is the size of a steel I beam that can safely carry the load shown in Fig. 224?

7. How large a steel I beam is required to carry the load shown in Fig. 209?

8. Investigate the security of the beam shown in Fig. 240. The material is cast iron.

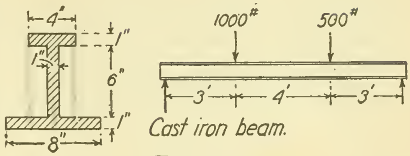


FIG. 240

9. A 12″ × 31 lbs. I beam, 16′ 0″ long, carries a distributed load which varies uniformly from zero at one end to a maximum at the other end. What is the total safe load?
10. A 10″ × 25 lbs. I beam, 17′ 0″ long, rests on two supports which are 12′ 0″ apart, and projects 5′ 0″ beyond one support. What is the safe uniformly distributed load?
11. Design a wooden beam to carry a total load of 1,500 lbs. on a span of 12′ 0″. The load is a distributed load varying uniformly from zero at each end to a maximum at the center.
12. A bar of aluminum is 36″ long and its cross section is a triangle whose base is 2″ and whose altitude is 3″. The apex of the triangle points downward. It is to act as a simple beam and carry a concentrated load at the center of the span. What is the greatest load that can be safely carried?
13. A Tee beam of cast iron has a section similar to Fig. 239. The horizontal flange is 9″ wide and the vertical stem is 8″ high. The metal in flange and stem is 1″ thick. The beam carries a uniformly distributed load of 5,000 lbs. on a span of 8′ 0″. What are the maximum unit stresses in tension and in compression?

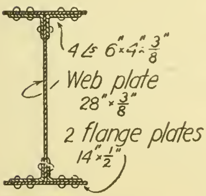


FIG. 241

14. Design a steel beam to carry a load of 3,000 lbs. concentrated at the center of a 12′ 0″ span.
15. Design a timber beam to serve the same purpose as in Problem 14.
16. Investigate the safety of a steel beam whose section is like Fig. 241 and which carries a load of 120,000 lbs. on a span of 30′ 0″. The intensity of the loading varies uniformly from zero pounds per foot at the right to a maximum at the left.
17. Given a beam whose cross section is like Fig. 197*A*, carrying a uniformly distributed load of 100,000 lbs. Determine the maximum unit stress.
18. Given a beam whose cross section is like that shown in Fig. 197*B*, and a span of 24′ 0″. Let the load vary uniformly from zero pounds per foot at each end to a maximum at the center. What is the *total* safe load?

137. Ultimate Bending Strength.—Modulus of Rupture. In § 135 the equation $M = s(I/c)$ is developed to express the relation between the loading on a beam and the unit stress produced by that load, in the outermost fiber of the beam. But in developing this relation, use was made of the proportionality between stress and deformation, and this proportionality exists only when the

stresses involved are less than the elastic limit of the material. Therefore, this equation may not be used to determine the unit stresses produced by loads which cause failure. Consider a beam on which the loads are gradually increasing. Let the left-hand

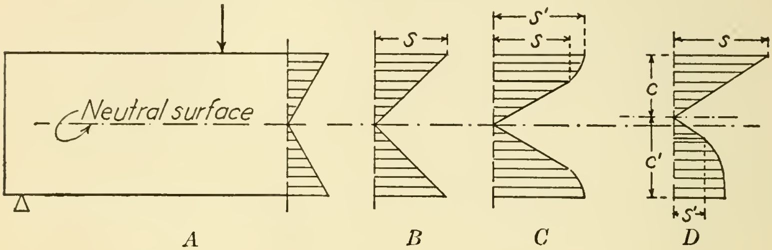


FIG. 242

part be shown by Fig. 242. At first the stresses are small and are distributed in the usual manner, as shown in A. As the load increases, the stresses increase, as shown in B, as long as the maximum unit stress s is within the elastic limit of the material. But when the elastic limit is exceeded, on the outermost fiber, the deformations *of that fiber* are greater in proportion to the unit stress than those on a fiber close to the neutral surface (§ 66). Hence the stress diagram must change its form to something like that shown in C. Here, s is the unit stress on the fiber which is stressed just to its elastic limit. Such a diagram will be symmetric about the neutral surface *only* in case the elastic limits of the material in tension and compression are equal and the beam section itself is symmetric about the same axis.

Now let us assume a beam with a symmetric section and made from some material having the same modulus of elasticity in tension and compression but a much higher elastic limit in compression than in tension. The stress distribution will be as shown in D. Here let s' be the elastic limit in tension while s is below the elastic limit in compression. From equation (3), § 133, we know that the areas of the stress distribution diagrams above and below the neutral surface must be equal. It follows that in the case assumed the neutral surface must be above the center of the section. That is, that $c' > c$.

It is thus apparent that under increasing loads (1) the form of the stress distribution diagram begins to change as soon as the elastic limit of the material is exceeded, and (2) if the section is unsymmetric in shape or strength, the neutral surface recedes from the weaker side as the unit stresses increase toward failure. While it is well recognized that these changes in stress distribution do occur, it is nevertheless common practice to determine what is known as the ***modulus of rupture*** by the use of equation (1), § 135. A beam is tested to destruction under known loads. The known loads, the span, and the section modulus are then inserted in the equation, and a value for s is worked out. This is the unit stress which *would* exist in the outermost fiber of the beam *if* the stress distribution diagram remained constant throughout the test. This value is a wholly fictitious one, but it has been found useful inasmuch as it has been found to be fairly consistent when derived from experiments on beams *of a similar material, size, and shape.*

Thus the *modulus of rupture* is a quantity which has no theoretical value but, when properly used, it is a useful measure of the ultimate strength of a beam. Its value is found to lie somewhere between the ultimate tensile and ultimate compressive strengths of the material. The values given in Table I under the heading "Bending" are derived in this manner.

138. Stress Distribution Diagrams. From equation (7), § 134, we find that the intensity of the unit stress s_1 at any point in a beam is $s_1 = Mz/I$. From this expression it is clear that in a beam of constant cross section s_1 varies with both M and z. It will have maximum values along the top or bottom face of the beam and at sections of maximum moment. It will be zero along the neutral surface and at sections of zero moment. Figure 250A is an approximate graphic representation of the variation of bending stress intensity throughout a beam of rectangular cross section, carrying a uniformly distributed load. In § 201 is a similar diagram to show the distribution of bending stress when combined with shearing stress.

PROBLEMS

1. What are the relative strengths of a pipe 4″ in diameter outside and $3\frac{1}{2}$″ in diameter inside, and a solid shaft 4″ in diameter? In each case the material is steel and it is used as a simple beam 8′ 0″ long to carry a uniformly distributed load.
2. What are the relative strengths of an 8″ × $17\frac{1}{2}$ lbs. I beam and a 10″ × 12″ wooden beam if each beam is 12′ 0″ long and is to carry a uniformly distributed load?
3. How much uniformly distributed load can be placed on a 24″ × 100 lbs. I beam 18′ 0″ long: (*a*) when the web of the beam is placed in a vertical plane? (*b*) When the web is horizontal?
4. How great a central load must be added to the loads on the beam in Fig. 240, in order to cause failure?
5. The modulus of rupture for a given material is 3,000 lbs. per sq. in. A beam 4″ × 8″ and 9′ 0″ long is made of this material. Two equal loads occur at 3′ 0″ and 6′ 0″ from the left end. How great may each load be if a factor of safety of 6 is to be allowed?

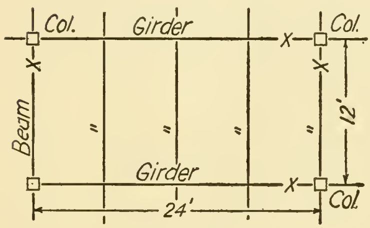


Fig. 243

6. What is the probable breaking load on a steel beam whose cross section is like Fig. 197*B* and whose span is 16′ 0″? The load is to be concentrated at the center.
7. A 20″ × 65 lbs. I beam, 20′ 0″ long, has a plate 7″ × $\frac{3}{4}$″ riveted to each flange. If no deduction is made for rivet holes, by what percentage is the strength of the plain I beam increased, when the plates are added to the section?
8. In Problem 7, if the rivets used are $\frac{7}{8}$″ rivets and two rivet holes are deducted from the area of *the tension flange only*, what is the percentage of added strength due to the plates?
9. A floor is to be built, supported on walls which are 12′ 0″ apart. Wooden beams, 2″ wide and 16″ on centers, are to be used. The flooring is to be of 2″ planks and the ceiling is to be of plaster, $\frac{3}{4}$″ thick. This floor must support, in addition to its own weight, a load of 50 lbs. per sq. ft. Design the beams.
10. Figure 243 shows a plan of a typical bay in a steel floor system. Similar bays surround it on all sides. The floor is intended to carry a total (dead and live) load of 250 lbs./sq. ft. What are the necessary sizes of I beams for girders and beams? The beams marked "x" rest directly on the columns.
11. In Problem 10, if the beam spacing is changed from 6′ 0″ to 8′ 0″, so that there are but two beams between the beams marked "x", will there be more or less steel required to support the floor? And how much?

CHAPTER XV

BEAMS—UNIT STRESS IN SHEAR

139. Introduction. In § 118 and especially in Fig. 200, the existence of vertical shear on a beam section was demonstrated. It is now necessary to determine how this shear is distributed over the cross section of a beam and to set up an equation to determine the *unit* shearing stresses throughout the section. In § 73 it was shown that whenever a shear exists in one direction, there is another *of equal intensity* and at right angles to the first. Thus if at any spot in a beam there is a vertical shear (as shown in § 118), then there must also be a horizontal shear (as shown in § 73). That such shear does exist can be easily perceived.

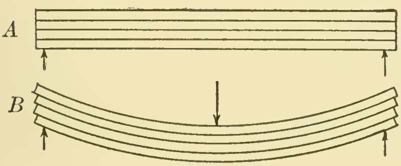


Fig. 244

Consider the beam shown in Fig. 244*A*, divided by imaginary horizontal planes as shown. Now imagine that these are actual cuts, offering no resistance to horizontal motion. As the beam deflects under its load, it will take on the shape shown in Fig. 244*B*, a shape which is familiar to anyone who has noted a pile of planks acting as a beam. The shape shown in Fig. 244*B* can result from that in 244*A* only by a horizontal slipping. If a solid beam does not take on the shape shown in *B*, it is due to its resistance to horizontal shearing stresses along the planes.

Another way of approaching the subject is shown in Fig. 245. Let *A* represent a beam loaded in any manner. Now let the shaded section be cut out and shown as a free body (at an enlarged scale) in *B*. The bending moment at section *AA* is less than at *BB*; therefore $s' < s$ and ΣH is not zero *unless* some stress H is present on the horizontal cut surface. If such a stress is applied, it will be seen that all of the conditions of equilibrium may be fulfilled; and not otherwise. On the other hand, if

$s = s'$ (that is, if there is no change of moment between AA and BB), neither V nor V' nor H can be present and still have equilibrium. Thus it is seen that shear always accompanies a *change* in moment and is not present otherwise. The same idea is stated in different terms in § 123. In § 140 we will develop an expres-

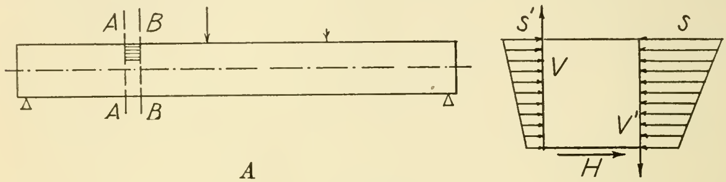


Fig. 245

sion for the shearing unit stresses by working chiefly with the horizontal shear. It should be remembered, however, that the intensities of the horizontal and vertical shears are equal and so the expression to be evolved will apply to either.

140. Unit Stress.—Shear. In Fig. 245A let the entire part of the beam between the planes AA and BB be cut out as a free

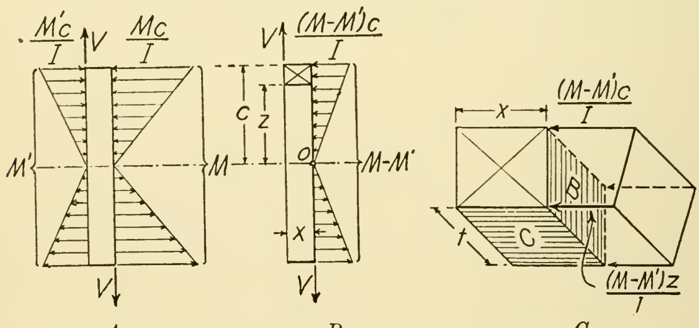


Fig. 246

body and shown (at an enlarged scale) in Fig. 246A. Let M be the resisting moment on plane BB and M' be that on AA. From the positions of the two planes, $M > M'$. Therefore the unit stresses on the right of the free body are greater than those on the left, as shown. Now from equation (1), § 135, the unit stresses on the outermost fibers on the two sides are Mc/I and

$M'c/I$. The shear is represented by the vertical arrows and its amount is V. Now let all the horizontal forces on the left of the free body be subtracted from those on the right and a new free body (Fig. 246B) be drawn. The moment stresses on this free body represent the stresses due to the *change* in moment between the two sections, i.e., $M - M'$. Taking a center of moments at o, we find $M - M' = Vx$, whence

$$V = \frac{M - M'}{x}. \tag{1}$$

This equation merely restates the fact that the shear is equal to the rate of change of the bending moment (see § 123). Now let a free body be cut from the one in Fig. 246B by a horizontal plane near the top face. Let this new free body be shown (at enlarged scale) in Fig. 246C. The unit stresses on the right-hand face vary from $(M - M')c/I$ to $(M - M')z/I$. The average unit stress will be

$$\left(\frac{M - M'}{I}\right)\left(\frac{c + z}{2}\right).$$

This unit stress is distributed over the area B; so that the total force acting horizontally on the free body is

$$\left(\frac{M - M'}{I}\right)\left(\frac{c + z}{2}\right)B.$$

This force tends to slide the free body horizontally. If sliding of this part does not occur *in the beam*, it must be because of resistance to sliding on the surface C, whose area is xt. Hence, if s_h denotes the average unit horizontal shearing stress on the area C,

$$s_h = \frac{\left(\frac{M - M'}{I}\right)\left(\frac{c + z}{2}\right)B}{xt}. \tag{2}$$

In this expression, $[(c + z)/2]B$ is recognized as the *static moment* (§ 53) of that part of the cross-sectional area of the beam which lies *above* the plane on which the unit shear is being determined, referred to the neutral surface. Let us call this quantity M_s. The rest of equation (2) can be simplified by substituting V for its equivalent value, as found in equation (1).

When these substitutions are made, equation (2) becomes

$$s_h = \frac{V}{It} M_s, \tag{3}$$

which gives the value of the horizontal (or vertical)* shearing unit stress at any point in a beam. Since (3) is derived by the use of (1), § 135, it is subject to the same limitations (§ 132).

141. Analysis of the Equation. Equation (3), § 140, easily can become a mere set of letters, leading to no real physical concepts, in which event it is worthless or worse. This condition may be avoided perhaps by going back to Fig. 246, and analyzing the equation in the light of the physical conditions there shown. The first fact of importance to be noted is that the shearing stresses are due to the *difference between* the moment stresses on any two vertical sections. Thus in Fig. 246*C*, the unit stress on the area *C* is equal to the force on the area *B* divided by the area *C*. Moreover the force on the area *B* depends on the average intensity of the moment stress and on the size of the area *B*. These relations may be expressed thus:

$$\left.\begin{array}{c}\text{Unit shearing stress}\\ \text{on the area } C\end{array}\right\} = \left\{\begin{array}{l}\text{The average moment stress on } B \text{ times}\\ \quad\text{the area } B\text{, divided by the area } C.\end{array}\right.$$

Equation (3), § 140, results from giving definite values to this simple statement. Let us take each member of (3) separately.

A. *As to M_s.* The symbol M_s stands for the static moment (referred to the neutral surface) of the area of that part of the section which lies above (or below)† the point on the section at which the required shearing stress occurs (§ 53). Its value will vary from zero (for a section taken at the top or bottom face of the beam) to a maximum when the section is taken at the neutral surface. Therefore the shearing unit stresses vary similarly.

B. *As to V.* Evidently the shearing unit stress should vary directly with the total shear, being generally large at sections near to the reactions.

* See § 139.

† Let the student prove that the static moment (referred to an axis through the center of gravity) of that part of an area which lies above a given point is equal numerically to that below the same point.

C. *As to I.* Since the moment stresses vary inversely with the I of the *entire section* (equation (1), § 135) and since the moment and shearing stresses are interdependent, it is but natural to expect to find I in the denominator of our equation.

D. *As to t.* Reference to Fig. 246C will show that the total push is distributed over an area C which varies with t. Hence the unit stresses naturally vary inversely as t.

E. *As to units.* Let the student show that the right-hand side of the equation is expressed in pounds per square inch.

142. Application of the Equation. (1) Let it be required to find the shearing unit stress in a beam 6″ × 8″ and 14′ 0″ long, carrying a uniform load of 500 lbs./ft., at a point 3′ 0″ from the reaction and at the neutral surface. The total shear at the given section is $(500 \times 14)/2 - (500)3 = 2{,}000$ lbs. The I for the beam is $6 \times 8 \times 8 \times 8/12 = 256$ (in.)^4. At the neutral surface M_s is

$$6 \times 4 \times 2 = 48 \text{ (in.)}^3.$$

The thickness of the beam is 6″. Therefore

$$s_h = \frac{2{,}000}{256 \times 6} 48 = 62.5 \quad \text{lbs./sq. in.}$$

In the above problem, at a point midway between the neutral surface and the top face $s_h = 46.9$ lbs./sq. in.

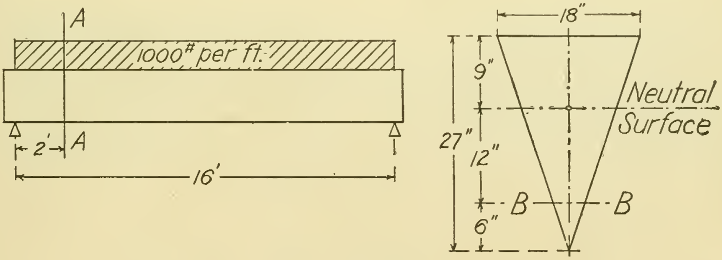


FIG. 247

(2) Let it be required to find the shearing unit stress in the beam shown in Fig. 247.

A. On the line of intersection of the neutral surface with

plane AA. The value of M_s can be conveniently figured *below* the neutral surface.

$$s_h = \frac{6{,}000}{\frac{18 \times (27)^3}{36} \times 12}\left(\frac{12 \times 18}{2}\right) \times 6 = 33 \quad \text{lbs./sq. in.}$$

B. On the line of intersection of planes AA and BB,

$$s_h = \frac{6{,}000}{\frac{18 \times (27)^3}{36} \times 4}\frac{(4 \times 6)}{2} \times 14 = 25.6 \quad \text{lbs./sq. in.}$$

C. In the case of an I beam (Fig. 254), the shearing unit stress may be determined as above by dividing the section into rectangles and triangles and neglecting the small curved fillets. A more detailed discussion of this case is given in § 204.

143. Stress Distribution Diagrams. From the above examples as well as from § 141, it is evident that the intensity of shearing unit stress varies widely at different heights on any given cross section. But since for any height whatever on a given section the total shear and the total I are constant, it is clear that the relative intensity of stress at various heights is determined by M_s/t (see equation (3), § 140). In the case of a rectangular cross section (Fig. 248) the intensity of the shearing unit stress on the plane AA (relatively to its intensity on some higher or lower plane) is therefore given by the expression

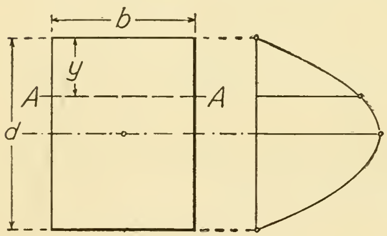


Fig. 248

$$s \text{ (relative)} = \frac{(by)\left(\frac{d}{2} - \frac{y}{2}\right)}{b} = \frac{yd}{2} - \frac{y^2}{2}.$$

This is the equation of a parabola. From this equation, the shearing unit stress has the value zero when y is zero and is a maximum when $y = d/2$, as shown in Fig. 248.

The *absolute* intensity of the maximum shearing stress on a rectangular section can easily be determined by direct substitution in equation (3), § 140. It is s_s (Max.) $= 3V/(2bd)$.

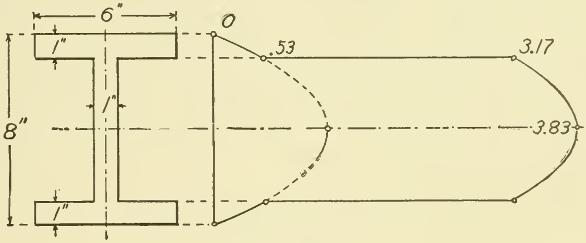
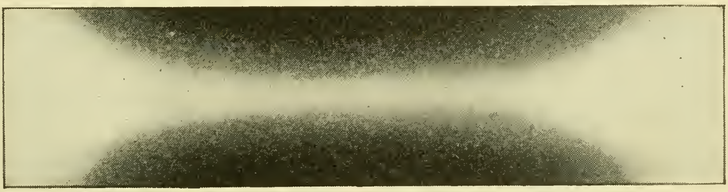
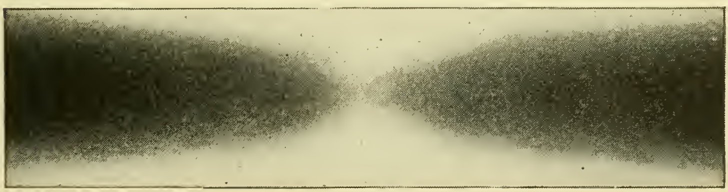


FIG. 249

In the case of an I section, Fig. 249, the stress distribution diagram was derived by actual computation. Let the student check the values. The relative stress intensities at a given height on various vertical sections through a beam will vary with the total shear on the section. Figure 250*B* is an attempt



A. Bending Moment



B. Shear

FIG. 250. Variation in Unit Stresses throughout a Beam of Rectangular Cross Section, carrying a Uniformly Distributed Load

to show the distribution of stress intensity as it varies both longitudinally and transversely in a rectangular beam carrying a uniformly distributed load. This diagram shows the intensity of *shearing* stress only. In § 201, Chapter XX is a diagram illustrating the combined effect of shear and bending stress.

PROBLEMS

1. A 2″ × 12″ wooden beam, 12′ 0″ long, carries a central load of 1,000 lbs. What is the factor of safety in shear?
2. A wooden beam, 4″ × 10″ and 10′ 0″ long, carries a uniformly distributed load of 3,500 lbs. What is the shearing unit stress at a point 3′ 6″ from one support and 3″ above the neutral surface?
3. A beam whose cross section is a triangle with a base of 2″ and an altitude of 3″ is used to carry a central load on a span of 36″. Let the apex of the triangular section point downward. If the allowable unit stress in shear is 3,000 lbs. per sq. in., what is the safe load as determined by the shearing resistance?
4. Investigate the beam in Problem 13, § 136, for shear.
5. What is the maximum shearing unit stress in the beam in Fig. 240?
6. Draw a diagram to show how the shearing unit stress varies from top to bottom on any cross section cut through each of the following beams: (*a*) a beam whose cross section is rectangular; (*b*) triangular; (*c*) an I shape similar to Fig. 347; (*d*) an angle shape similar to Fig. 194.
7. A wooden beam 6″ × 6″ and 6′ 0″ long is loaded to its full safe load in bending. What is the factor of safety in shear?
8. Given a steel beam whose section is like Fig. 347 and which spans 8′ 0″. Let it carry two loads each of which is 100,000 lbs. and placed 2′ 0″ from a support. What is the factor of safety in shear?

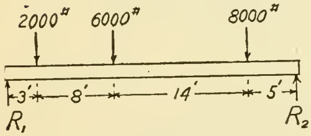


Fig. 251

9. In Problem 8, what is the shearing unit stress at a point near the end of the beam and just below the junction of web and flange? Also at a point near the end of the beam and just above the junction of web and flange?
10. A 24″ × 80 lb. I beam, 7′ 0″ long, carries a uniformly distributed load of 240,000 lbs. Draw a diagram showing the intensity of the shearing unit stress throughout a section cut close to one support.
11. A wooden beam 16″ × 16″ is used to carry the loads shown in Fig. 251. What are the maximum unit stresses in bending and in shear?
12. A beam is made up of two pieces of timber, 2″ × 2″ and 6′ 0″ long, glued together. It carries a concentrated load of 200 lbs. at its center. What is the maximum stress per square inch on the glued joint? The joint lies in a horizontal plane.

CHAPTER XVI

BEAMS—CHARACTERISTIC SHAPES AND RELATIONS

144. Introduction. The beams used in structural work are mostly of wood, steel, cast iron, or reinforced concrete. Wooden beams are sawed from logs. Steel beams are formed by passing heated billets through rollers which give the required shape. Both of these processes produce beams of uniform cross section throughout the length. Because of this fact and the ease with which beams of uniform cross section can be fitted together and incorporated in a structure, a very large proportion of all the beams used in structural work are simple prisms. On the other hand, cast iron and concrete are moulded into shape and, as a matter of manufacture, it is much less difficult to vary the section.

In Chapters XIV and XV we have seen how the bending moment and shear vary along the length of a beam and how the resisting stresses vary on a given cross section. Economy of *material alone* would dictate the forming of beams in such a manner as to produce unit stresses equal to the working strength of the material on the outermost fiber of every cross section, and so to dispose the area that as few fibers as possible occur near the neutral surface. In a general sort of way this would give a beam, deeper in the center than at the ends (lengthwise) and wider at the top and bottom faces than toward the center (of the cross section). In most cases the labor of producing such a beam is excessive and more than offsets any possible economy of material. But occasionally there are conditions which justify a considerable variation from the simple forms ordinarily used. Where loads or spans are so large as to call for sizes in excess of the largest standard rolled section, the problem becomes a special one and important economies may be effected by careful design. Again when a material, such as cast iron, has widely differing strengths in tension and com-

pression, it is possible to adjust the form of cross section to this fact.

145. Rectangular Sections. A. Proportion of Depth to Width of Cross Section. The resisting moment of any beam varies as its section modulus (§ 135). The section modulus (I/c) for a rectangle is $bd^2/6$. From this expression it will be seen that the strength in bending of a beam of rectangular section varies directly with the width and with the square of the depth. The quantity of material in the beam, however, varies with the area of cross section (bd). Thus the most economical proportion is that one for which bd^2 is a maximum for a given area. Evidently this is approached as b decreases and d increases without limit. For example a square section 3.46″ on a side has an area of 12 sq. in. and the factor bd^2 amounts to 41.6. For a rectangle 3″ × 4″, bd^2 is 48; for a 2″ × 6″ section, 72, and for a 1″ × 12″ section it is 144. Thus the *relative* bending strengths of beams containing the same amount of material are seen to vary through a wide range.

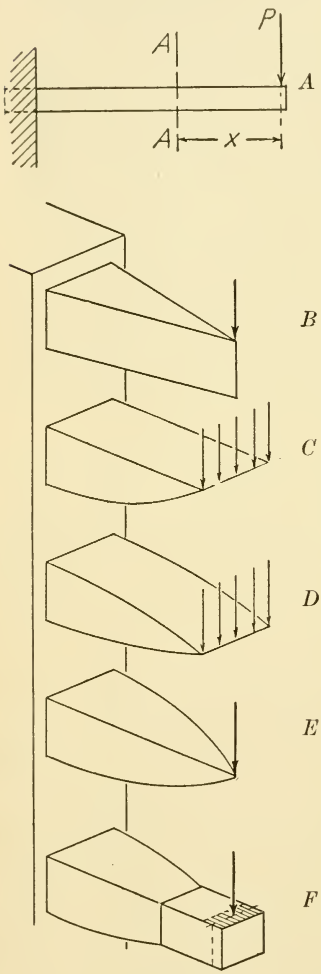


Fig. 252

In § 143, it is shown that the strength of a beam of rectangular section, in shear, varies with the cross-sectional area alone.

We may then conclude that for beams of rectangular section a narrow deep proportion is best. In the case of wooden beams, the minimum width is established by nailing and other conditions

having to do with manufacture, transport, erection, and fire resistance.

B. Beam of Uniform Strength. In the case of a cantilever beam with a load concentrated at the end (Fig. 252*A*), the bending moment at any section A varies with x. Since the resisting moment of the rectangular section varies with bd^2, it will be evident that in a beam whose b varies with x while d is a constant, there will be constant relation between the bending moment and the resisting moment, i.e., the unit stress on the outermost fiber is the same for any section. Such a beam (Fig. 252*B*) is called a beam of uniform strength. The constant relation between bending and resisting moments may be secured by varying other factors, as shown in Figs. 252*C*, *D*, and *E*.

In the above cases, the strength in shear is not considered. The shear at the end of the beam would call for a modification of shape somewhat as shown in Fig. 252*F*.

C. Strongest Beam that can be cut from a Log. This problem requires us to find that rectangle which, being inscribed in a circle of a given size, has the maximum possible value of bd^2. From Fig. 253, $d^2 = D^2 - b^2$. We then need to find that proportion of b and d which will give to the expression $b(D^2 - b^2)$ a maximum value. By the usual process of the differential calculus this is found to occur when

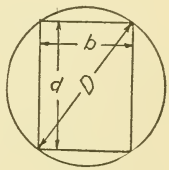


Fig. 253

$$b = D\sqrt{\tfrac{1}{3}} \qquad \text{and} \qquad d = D\sqrt{\tfrac{2}{3}} = \sqrt{2}\,b = 1.41b.$$

D. Relation of Span and Depth. In very long beams, a small load produces a large moment and a small shear. In very short beams, a large load produces a large shear and a small moment. It is evident that at some span, the shear and moment are equally important.

Let us take the case of a rectangular beam carrying a uniformly distributed load. Let the allowable fiber stress in bending be s and in shear s_s. Then the bending moment is $WL/8$, and the resisting moment is $sbd^2/6$. From these two values, the allow-

able load, as determined by bending, is found to be $4sbd^2/(3L)$. Again from equation (1), § 143, the allowable load, as determined by shear, is found to be $4s_sbd/3$. In the critical case above mentioned these two loads are equal; this gives

$$\frac{4}{3}\frac{sbd^2}{L} = \frac{4}{3}s_sbd,$$

or

$$L = \frac{s}{s_s}d, \tag{1}$$

which is the ratio of depth to span at which the degree of security is the same in bending and in shear. Other ratios can easily be worked out for other conditions of loading.

PROBLEMS

NOTE. In problems 1–6, let bending moment only be considered.

1. Draw four different simple beams of uniform strength to carry a load concentrated at the center.
2. Draw two different simple beams of uniform strength to carry a uniformly distributed load.
3. A wooden beam is to be used as a cantilever 5′ 0″ long and is to carry a uniformly distributed load of 1,200 lbs. Design a beam of uniform strength for this purpose.
4. Two posts are 11′ 0″ apart. A wooden beam 21′ 0″ long spans between the posts and overhangs 5′ 0″ on each side. The beam is 4″ broad and carries a uniformly distributed load of 6,300 lbs. Design a beam of uniform strength for this purpose.
5. A wooden beam spans 14′ 0″ between supports. At 4′ 0″ from the left is a concentrated load of 1,000 lbs. and 5′ 0″ from that load is another of 1,800 lbs. Design a beam of uniform strength for this service.
6. Design a beam of rectangular cross section and uniform strength to span 10′ 0″ and carry a load which varies from zero at one end to 1,000 lbs. per ft. at the other end.
7. A certain wood has a safe bending strength of 1,000 lbs. per sq. in. and a safe shearing strength of 70 lbs. per sq. in. At what span will a 6″ × 6″ beam of the above wood which is loaded to its full load in bending be also stressed to its full capacity in shear?
8. What is the greatest safe load that may be placed on an 8″ × 12″ timber beam?

146. Rolled Sections. A rolled beam is, from the method of manufacture, necessarily of a constant cross section. But this

section can easily be given a wide variety of shapes. The shape most commonly used for beams is the I section shown in Fig. 254. These sections are rolled in a variety of depths and several weights for each depth as listed in any of the various handbooks. The precise details, such as slope of flanges, radii of fillets, etc., have been determined by the processes of manufacture, ease in fabrication, and other such considerations, but the general form is the result of an attempt to so dispose the metal as to give a high resistance to bending. Hence the large concentration of metal in the flanges to increase the moment of inertia of the section. The web is kept as thin as possible while still retaining enough metal to give satisfactory shear resistance between the flanges (see § 143). Other rolled sections may be used as beams, but if so it is done as a matter of convenience rather than because of their efficiency in carrying load.

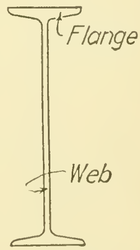


FIG. 254

147. Unsymmetric Sections. In a symmetric cross section, the unit stresses on the uppermost and lowermost fibers are equal. Hence this form of section is well adapted to a material such as steel, which has about the same unit strengths in tension and in compression.

In the case of a material like wood, which has a greater strength in tension than in compression, the upper side of a symmetric section will fail first. In a stone or a plain concrete beam of symmetric section, the reverse is true. For such materials, then, it would seem that some form of section that would place a greater amount of material on the weak side of the neutral surface would prove advantageous. One form of beam section, designed to meet this condition, is the Tee section shown in Fig. 255. Here the neutral surface is well down toward the bottom and, in a beam of this section, the greatest unit stress on the compressive side would be nearly

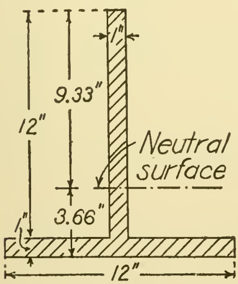


FIG. 255

three times as large as that on the tensile side (equation (7), § 134). This fact would indicate the use of such sections for materials stronger in compression than in tension. As a matter of fact, cast iron is frequently used in this form. On the other hand, such materials as wood and stone cannot readily be worked to such shapes and so are more commonly seen with rectangular sections even though there is a loss of strength per unit of material. In the case of reinforced concrete beams the visible sections are usually symmetric, but the lack of strength on the tensile side is made up by introducing a stronger material (steel), as shown in Fig. 378. This produces the same effect as spreading out the area on the lower side (Fig. 369). This case is more fully covered in §§ 210–216.

148. Plate Girders. For very great spans or loads, it is usual to build up steel girders formed of various sections united by riveting. In such a case it is possible to approximate a beam of uniform strength quite closely. A typical form of plate girder is shown in Fig. 256. The flanges are increased in area

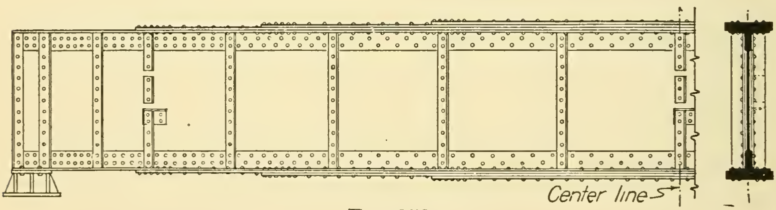


Fig. 256

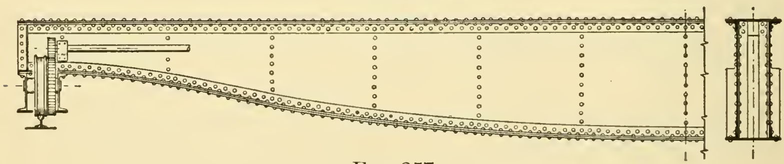


Fig. 257

toward the center, thus providing for the bending stresses. The web is stiffened toward the supports, thus providing for the heavier shear which occurs there. The riveting which unites the various parts must transmit horizontal shear and consequently is more closely spaced near to the supports.

Another form of girder is shown in Fig. 257. In this case the depth is varied rather than the flange area. This form is frequently used for traveling cranes.

PROBLEMS

1. What are the relative strengths of the lightest and the heaviest 18″ I beams listed in the handbooks?
2. Given an 18″ × 55 lb. I beam and a beam of solid steel, 6″ × 18″, what are their relative strengths per pound of metal?
3. Two plates each 6″ × ½″ are to be riveted to a 12″ × 31½ lb. I beam. Rivet holes are not to be considered. By what percentage will the strength of the plain beam be increased (*a*) if the plates are riveted to the flanges, (*b*) if they are riveted each side of the center of the web?
4. What are the relative strengths per pound of metal of a 4″ × 4″ × ½″ T beam (flange horizontal and stem pointing up) and a 6″ × 12¼ lb. I beam?
5. What are the relative strengths of a 4″ × 4″ × ½″ T beam (flange horizontal and stem pointing up) when made of steel and when made of cast iron?

NOTE. The modulus of rupture for cast iron as given in Table I was derived from experiments on small beams of rectangular section and therefore should not be used in this problem (§ 137). The ultimate strengths in tension and compression should be used.

6. What are the relative strengths of the cast iron T beam in Problem 5, when the stem points up and when it points down?

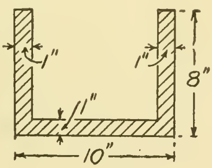


FIG. 258

FIG. 259

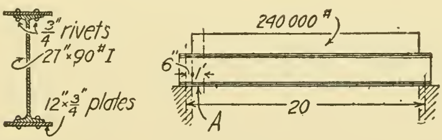


FIG. 260

7. How much uniformly distributed load may be safely carried on a cast iron beam 9′ 0″ long? The section of the beam is like Fig. 258. (See note to Problem 5.)

8. How much uniformly distributed load may be placed on two 15″ × 42 lb. I beams: (*a*) when placed side by side; (*b*) when superimposed and riveted together as shown in Fig. 259? The span in each case is 25′ 0″.
9. Figure 260 shows a riveted beam girder: (*a*) What is the maximum bending unit stress? (*b*) How many rivets are needed in the space *A*?
10. What is the safe uniformly distributed load on the plate girder shown in Fig. 241, as determined by bending on a span of 30′?
11. Draw a typical girder, similar to Fig. 257, to carry two symmetrical concentrated loads at the third points of the span.

CHAPTER XVII

BEAMS—DEFORMATION

149. Introduction. The deformations produced in a beam are principally those due to the bending moment.* Compressive stresses shorten the upper fibers of the beam while the tensile stresses elongate the lower fibers. Due to these different changes in length in different parts, the beam is forced into a curved shape, as noted in § 117.

In Fig. 261 is shown a beam, deformed by the load W. The top fiber aa is shorter, and the bottom bb is longer than before bending occurred, while the neutral surface cc remains at its original length L. The end planes ab tip inward as necessitated by these conditions, and the center of the beam is depressed below the level of the supports. The amount of this depression y is called the *deflection.*

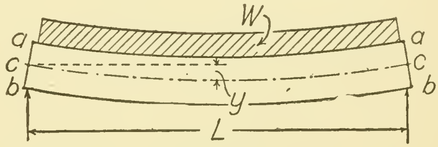


Fig. 261

A careful study of the phenomena of deflection is important, since in many cases excessive deflection is very undesirable. For instance, when shafting must be kept aligned or when ceilings of plaster or similar materials are to be carried, excessive *deflection* can cause serious trouble even in a case where ample *strength* is provided. Again, from the theoretical standpoint, the study of beam deformations is most important as it leads directly to the solution of many statically indeterminate problems such as beams on three or more supports, stresses in columns, etc.

* The shearing stresses which are present in a beam produce a small amount of deflection. Extended works on strength of materials give methods for its evaluation. But, except in very deep and short beams, the amount of deflection due to shear is so small a percentage of the total deflection that it may be neglected. Therefore the following discussion is concerned only with deformations due to bending moment stresses.

Since it is the bending stresses that produce the deformations, and the deformations finally produce the deflection, it is natural to expect that in any expression for deflection there will be found about the same factors as in the expression for strength in bending. This is actually the case. The general form of the expression for deflection is

$$y = \text{a function of } \frac{WL^3}{EI}.$$

In this expression L and W represent the effect of span and loading, I represents the moment of inertia of the section as determined by its size and shape, and E represents the stiffness of the material of which the beam is made.

The curve formed by the neutral surface of a beam is called the ***elastic curve*** (as shown by *cc*, Fig. 261). In developing an expression for deflection, frequent use will be made of (1), § 135. Therefore the limitations governing that equation (§ 132) will apply equally to all results obtained for deflections.

150. General Form of the Elastic Curve. Let *abcd*, Fig. 262*A*, represent a beam and let it be divided into small equal divisions by planes normal to its longitudinal axis, as g–h, i–j, etc. Let *ef* be the neutral surface.

Now let forces P be applied so as to subject the beam to a bending moment equal to Pr. This bending moment is the same on every cross section of the beam (§ 127) and will produce equal deformations in each of the small sections, a, g, h, c; g, h, j, i; etc. (The deformations in the figure are greatly exaggerated.)

From the principle of the maintenance of plane sections (§ 130), the deformed shape of section k, l, n, m is seen to be the trapezoidal shape k', l', n', m'; and this same change of shape takes place in each of the small sections comprising the beam because in each section it is the result of the action of the same bending moment. It is evident then that the beam will take the form of a regular polygon, composed of the equal trapezoids A, B, C, D, Fig. 262*B*. Here again the deformations are greatly exaggerated. If now the number of the sections considered is

allowed to increase indefinitely, the broken line *ef* approaches a circular curve which is the elastic curve of the beam.

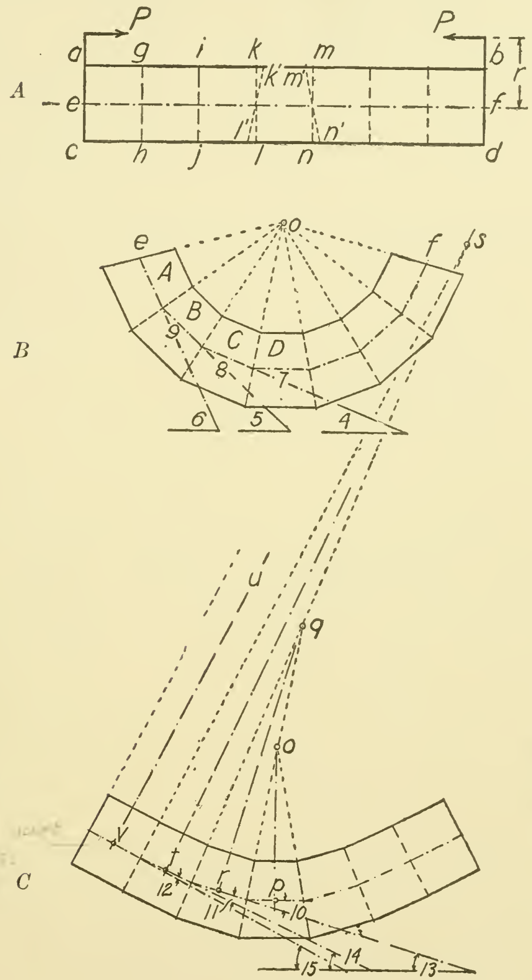


FIG. 262

It should be particularly noted that the angles *4*, *5*, and *6*, Fig. 262*B*, define the *slope* of the elastic curve through sections *C*, *B*, and *A*, respectively. Now the *difference* between these slopes which occurs in one section (the *rate of change* of the slope)

is represented by any of the angles *7*, *8*, or *9*. Since we are dealing with a regular polygon, these angles are equal and the absolute size of any angle depends directly on the bending moment which causes the deformation.

Moreover, the equality of the angles *7*, *8*, and *9* indicates that in this case the elastic curve is a circle. Thus a uniform bending moment is seen to produce a circular form in the elastic curve.

Let us now pass to a case where the bending moment varies, being greatest at the center, and zero at the ends, as in Fig. 262C. In this case the center sections will suffer the greatest deformations, and the end sections the least. The elastic curve will have no curvature at the ends and a comparatively large curvature at the middle. The radius of curvature at the center will be op; at the next section qr, etc., until at the end, it is infinite.

The slope of the elastic curve, as defined by the angles *13–14–15*, is seen to increase toward the end, but the *rate* of increase, as shown by the angles *10*, *11*, *12*, diminishes (as the moment diminishes) toward the end. This principle is the fundamental one governing the deformation of beams. Putting it in a slightly different way, we may say that it is the bending stresses that produce the change in the shape of the elementary slices from rectangular to trapezoidal (Fig. 262A). When the change in any elementary slice is great, the change in the slope of the elastic curve, at that slice, will be great. Thus it is seen at once that the *amount* of the bending moment governs the *rate of change* of the slope of the elastic curve.

151. Form of the Elastic Curve.—By Inspection. The principle developed in § 150 may be used to determine, by inspection, the general character of the elastic curve for any given loading.

In Fig. 263A, a simple beam is shown with a load concentrated at its center. The bending moment diagram is shown in B. Now let the line aa in C be taken as a base from which to draw a curve whose ordinates will display the *variation* in the slope of the elastic curve D.* From symmetry we know that the elastic curve will be horizontal (no slope) at the center. Therefore our

* That is, each ordinate of curve C gives the amount of the slope of curve D at the corresponding abscissa. The curve C, taken as a whole, shows the *variation* in the slope of curve D.

curve will pass through b. Moreover the rate of change of the y ordinate at this point (as shown by the bending moment, § 150) will be a maximum; indicating a steep slope to the curve in question. At the ends our curve will be horizontal, as the

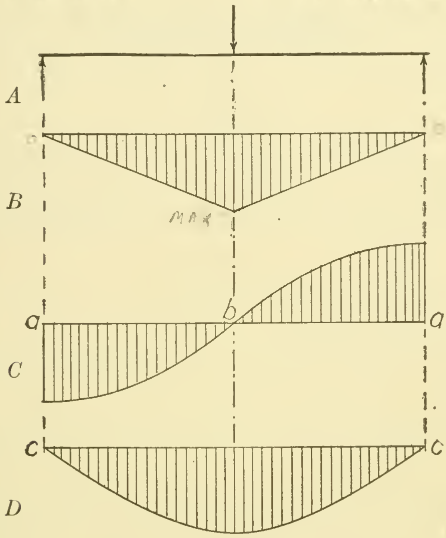


Fig. 263

bending moment is zero. But we know that the *elastic curve itself* must have a maximum *slope* at the ends (Fig. 262C) and that the slope at the left is opposite in sense to that at the right. These considerations produce the slope curve as shown in C.

Now let the line cc in D be used as a base line to draw the elastic curve itself. As shown by its slope curve, it is horizontal at the center, steepest at the ends, and changes curvature more rapidly as it nears the center.* The radii of curvature increase from the center towards the ends (Fig. 262C).

* The full significance of this initial slope at the ends of the beam will perhaps not be apparent until we reach the discussion concerning the "constant of integration" in § 154, where its value is definitely determined. However for the present it is possible to see, from purely physical considerations, that the slope of the elastic curve does have its maximum value at the support.

Obviously any such attempt to trace out slope and deflection curves must be subject to limitations and cannot give quanti-

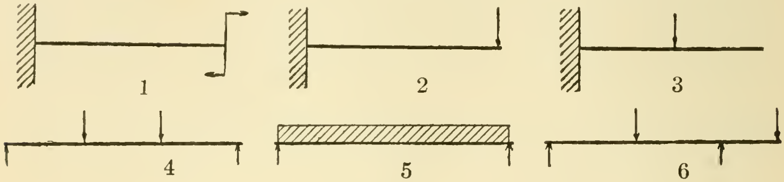


Fig. 264

tative results. However, before starting to develop definite values for deflections it will be worth while for the student to trace out similar curves for the cases of loading shown in Fig. 264.

PROBLEMS

Draw by inspection the general form of the slope curve and the elastic curve in each of the following cases:

(1) Simple beam with two unsymmetric concentrated loads.

(2) Simple beam with uniformly varying load.

(3) Cantilever beam with concentrated loads at end and center.

(4) Beam overhanging both ends and with uniformly distributed load.

152. General Equation of the Elastic Curve. Our object is now to work out an equation for the elastic curve so that we can evaluate the departure of the curve from a straight line (the deflection of the beam) in definite, measurable terms.

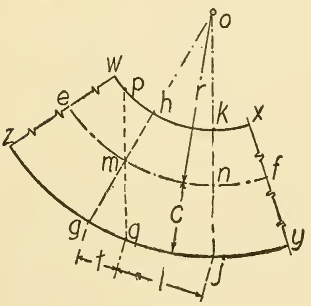


Fig. 265

In Fig. 265, let *wxyz* represent a portion of a bent beam, whose neutral surface is *ef*. Let *ghkj* be a small part of the beam included between two plane sections which, before the beam was bent, were parallel. Moreover let *nm* be conceived to represent so small a part of the neutral surface that the curve between these points can be regarded as a circular curve of radius *r*, and that the distance *nm* can be regarded, *without sensible error*, as being of the same length along the curve or along a straight line between the planes *go*, *jo*, before bending took place,

Now let qp (parallel to kj) represent the position of gh before bending occurred. Then evidently qg represents the deformation t which has taken place in the outermost fibers whose original length was l.

Now if s denotes the unit stress on the outermost fibers and E denotes the modulus of elasticity of the material, then

$$(1) \qquad E = \frac{s}{\frac{t}{l}}, \qquad \text{or} \qquad \frac{l}{t} = \frac{E}{s}.$$

But the triangles mno and gqm are similar; hence

$$(2) \qquad \frac{r}{c} = \frac{mn}{t} \qquad \text{or} \qquad \frac{r}{c} = \frac{l}{t}.$$

Now substituting the value of l/t from (2) in (1), we find

$$(3) \qquad \frac{r}{c} = \frac{E}{s}, \qquad \text{or} \qquad r = \frac{Ec}{s}.$$

Now taking the expression for stress due to bending (§ 135) $s = Mc/I$, and substituting this value of s in (3), we get

$$(4) \qquad r = \frac{EI}{M}, \qquad \text{or} \qquad \frac{1}{r} = \frac{M}{EI}.$$

This expression is a direct relation between the radius of curvature of the elastic curve of a beam *at any point of its length* and the bending moment *at the same section.*

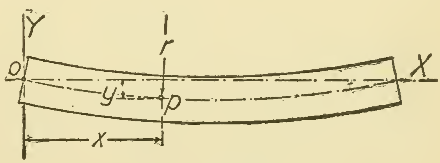


FIG. 266

Before proceeding further let the student check up on the following queries with regard to equation (4):

(*a*) In what units will each side be expressed?

(*b*) Why does each factor occur where it does?

(*c*) Does it check with the principles laid down in § 150?

Our next problem is seen by reference to Fig. 266. Here the point p is on the elastic curve of a bent beam. Let OX, OY be coordinate axes for the elastic curve. Then the point p is the point (x, y); y is the deflection of the beam at the point p; and r is the radius of curvature of the elastic curve at p. Now from the calculus we know for any curve that

$$(5) \qquad \frac{1}{r} = \frac{\dfrac{d^2y}{dx^2}}{\left(1 + \left(\dfrac{dy}{dx}\right)^2\right)^{3/2}}.$$

But the curves with which we are concerned are very flat and hence dy/dx (the slope of the curve at the point xy) is small compared with unity. Hence it is assumed that the powers of dy/dx can be neglected without sensible error. Then (5) becomes

$$(6) \qquad \frac{1}{r} = \frac{d^2y}{dx^2}.$$

Equating the values of $1/r$ in (4) and (6), we have

$$(7) \qquad \frac{d^2y}{dx^2} = \frac{M}{EI}.$$

This is the general equation of the elastic curve of any beam.

153. Significance of the Equation. When a curve is expressed by an equation between x and y, dy/dx is the slope of the curve at the point xy. The rate of change of the slope is

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) \quad \text{or} \quad \frac{d^2y}{dx^2}.$$

Thus equation (7), § 152, is seen to express the same principle that was developed in § 150, viz., that the bending moment measures the rate of change of the slope of the elastic curve.

In most beams the material and the cross section are constant throughout the length. In what follows it will be assumed that this is the case.* On the other hand, M is usually expressible

* Special means for handling the deflection of beams of variable cross section may be devised if I is expressible as a function of x.

as a function of x. Therefore equation (7)
sidered as in the form

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \phi(x).$$

It is then evident that one integration will
i.e., the slope of the elastic curve, and that
will give the equation of the elastic curve
equation the values of y (the deflection) can
given values of x. Now for convenience let the
dy/dx be expressed by v. Then equation (7), §

$$\text{(8)} \qquad \frac{dv}{dx} = \frac{M}{EI}, \qquad \text{or} \qquad EIdv = Mdx.$$

Now integrating this equation, we get

$$\text{(9)} \qquad EIv = \int Mdx,$$

which is the equation for a curve whose ordinates express the *slope* of the elastic curve. Substituting dy/dx for its value v, we obtain

$$EI\frac{dy}{dx} = EIv.$$

Integrating this expression, we get

$$\text{(10)} \qquad EIy = \int EIvdx,$$

which is the equation of the elastic curve itself. From this equation, values for the deflection may be obtained.

154. Application of the General Equation. In order to determine the deflection of a beam of uniform material and cross section, it is evident from what precedes that we must know E and I and that we must be able to express M in terms of x. Then by equation (9) above we can derive the slope curve, and by equation (10) we can get the required deflection.

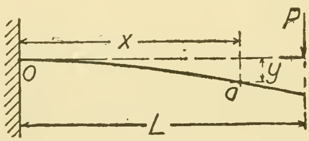


FIG. 267

As an example, take a cantilever beam loaded at the end

it be required to find the deflection at the
ıates are (x, y), the origin being taken at O.
ling moment is negative (§ 122), and its
$-x)$. Inserting this value in equation (9),
slope of the elastic curve at the point a is

$$- P(L - x)dx = - P \int (L - x)dx$$

$$- P\left(Lx - \frac{x^2}{2}\right) + C.$$

ng expression, the constant of integration C is the elastic curve at the support, as discussed in § 151. the slope has the value zero when x is zero. There- 0, and the above equation becomes

$$EIv = - P\left(Lx - \frac{x^2}{2}\right), \quad \text{(11)}$$

which gives the *slope* of the elastic curve at a. Let the student assure himself that, in the above equation, v is actually a mere ratio.

To determine the *deflection* at the point a, take the value of EIv in equation (11) and substitute it in equation (10), § 153. We find

$$EIy = \int - P\left(Lx - \frac{x^2}{2}\right)dx = - P\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) + C'.$$

To determine C', we know that $y = 0$ when $x = 0$. Therefore $C' = 0$, and the preceding equation becomes

$$y = - \frac{P}{EI}\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right). \quad \text{(12)}$$

From the physical limitations of the problem, we can see that the greatest values for slope and deflection occur at the end of the beam, where $x = L$. By substitution in (11) and (12) these values are found to be

$$v \max = - \frac{PL^2}{2EI}, \qquad y \max = - \frac{PL^3}{3EI}.$$

155. Further Applications. The following cases are worked out in the same order as that in § 154, but in a condensed form. Their chief use is to give opportunity for comparison and to bring out the significance of the constant of integration.

A. *Cantilever beam. Uniformly distributed load of w lb. per ft.* (Fig. 268*A*). The bending moment at a section distant x from the support (see Fig. 268*B*) is

$$M = -w(L-x)\frac{(L-x)}{2} = -\frac{w}{2}(L-x)^2,$$

$$EIv = -\frac{w}{2}\int(L-x)^2dx = -\frac{w}{2}\left(L^2x - Lx^2 + \frac{x^3}{3}\right) + C.$$

But $v = 0$ when $x = 0$. Therefore $C = 0$, and (see Fig. 268*C*)

$$EIv = -\frac{w}{2}\left(L^2x - Lx^2 + \frac{x^3}{3}\right).$$

Again

$$EIy = -\frac{w}{2}\int\left(L^2x - Lx^2 + \frac{x^3}{3}\right)dx$$

$$= -\frac{w}{2}\left(\frac{L^2x^2}{2} - \frac{Lx^3}{3} + \frac{x^4}{12}\right) + C'.$$

But $y = 0$ when $x = 0$. Therefore $C' = 0$, and (see Fig. 268*D*)

$$y = -\frac{w}{2EI}\left(\frac{L^2x^2}{2} - \frac{Lx^3}{3} + \frac{x^4}{12}\right).$$

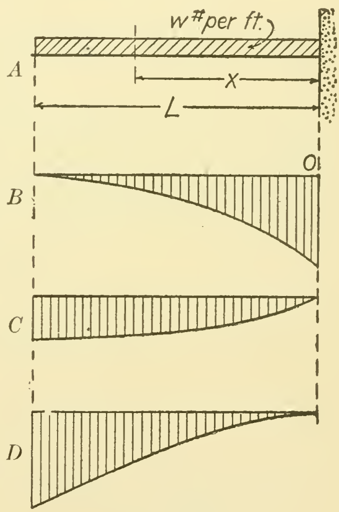


FIG. 268

The greatest slope and deflection occur at the end of the beam where $x = L$; their values are

$$v \max = -\frac{wL^3}{6EI} = -\frac{WL^2}{6EI},$$

$$y \max = -\frac{wL^4}{8EI} = -\frac{WL^3}{8EI}.$$

B. *Simple beam. Load concentrated at the center* (Fig. 263). The bending moment, slope, and deflection curves have different equations on the two sides of the center. We will develop those for the left half, the right half being symmetric with the left. Taking the origin for the slope curve at the left support, we have $M = (P/2)x$, and

$$EIv = \int \frac{P}{2} x\, dx = \frac{P}{2} \int x\, dx = \frac{P}{4} x^2 + C.$$

Here $v = 0$ when $x = L/2$; hence $C = -PL^2/16$, and

$$EIv = \frac{P}{4}\left(x^2 - \frac{L^2}{4}\right).$$

For the deflection we have

$$EIy = \int \frac{P}{4}\left(x^2 - \frac{L^2}{4}\right) dx = \frac{P}{4}\left(\frac{x^3}{3} - \frac{L^2 x}{4}\right) + C'.$$

Here $y = 0$ when $x = 0$; hence $C' = 0$, and

$$y = \frac{P}{4EI}\left(\frac{x^3}{3} - \frac{L^2 x}{4}\right).$$

The greatest slope is at the support; the greatest deflection at the center; their values are

$$v \max = -\frac{PL^2}{16EI},$$

$$y \max = -\frac{PL^3}{48EI}.$$

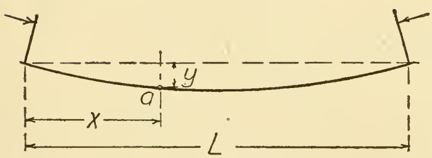


Fig. 269

C. *Simple beam with uniform moment.* Let Fig. 269 represent a simple beam bent by forces producing a uniform moment equal to M. Then the slope of the elastic curve at any point xy is given by

$$EIv = \int M dx = Mx + C.$$

Equation of curve $f(x) = \frac{W}{2EI}\left(\frac{Lx^2}{2} - \frac{x^3}{3} - \frac{L^3 x}{12}\right)$

Sub. $\frac{L}{2}$ you get $\frac{5WL^3}{384EI}$

In this case $v = 0$ when $x = L/2$. Therefore $C = -ML/2$, and

$$EIv = M\left(x - \frac{L}{2}\right).$$

For deflection

$$EIy = \int EIv\,dx = \int M\left(x - \frac{L}{2}\right)dx = \frac{M}{2}(x^2 - Lx) + C'.$$

But $y = 0$ when $x = 0$. Therefore $C' = 0$, and

$$EIy = \frac{M}{2}(x^2 - Lx).$$

The greatest slope occurs where $x = 0$. The greatest deflection occurs where $x = L/2$. Therefore

$$v \max = -\frac{ML}{2EI},$$

$$y \max = -\frac{ML^2}{8EI}.$$

D. *Other cases.* In Table III of the Appendix will be found the slope and deflection curves for a number of typical cases of loading. These should be studied carefully with reference to the relations between all five curves, as outlined in § 156.

PROBLEMS

Derive the equations for the slope and elastic curves and the value of the maximum deflection in each of the following cases. Use standard notation.

(1) Cantilever beam under uniform moment of Pa lb. ins.

√ (2) Simple beam with uniformly distributed load.

156. Comparison of Curves. Table III, in the Appendix, gives curves for loading, shear, moment, slope, and deflection in a number of typical cases. The relations between these curves are interesting and will serve to unify the entire study of beams.

In § 121, it was shown that the loading governs the rate of change of the shear. In § 123, we saw that the shear governs the rate of change of the bending moment. In § 150 and equa-

tion (8), § 153, we saw that the moment governs the rate of change of the slope of the elastic curve. Obviously the slope governs the rate of change of the ordinates of the elastic curve (the deflection).

Again, when such a set of relations prevail, the ordinate on any curve is determined by the area under the curve next above it.*

In studying these relations as shown by the curves in Table III, due allowance must be made for the fact that reactions are, in effect, a part of the loading, and give to the shear an initial value, which corresponds to the constant of integration. Moreover, in the slope curve we have a definite slope in the beam at the left reaction; this also is the constant of integration, as was pointed out in § 154 and in the footnote on page 211. In both of these cases the area under the curve above corresponds to the ordinate, *measured from the dotted line.* The position of this line is fixed by the constant of integration which determines the initial value of the ordinate for the curve.

These curves and the relations between them deserve to be studied carefully, until the student can sense the relations visually and can readily account for the way in which powers of x ascend as we pass from the load curve to the deflection curve and also how the powers of x ascend from the simpler cases of loading to the more complex cases.

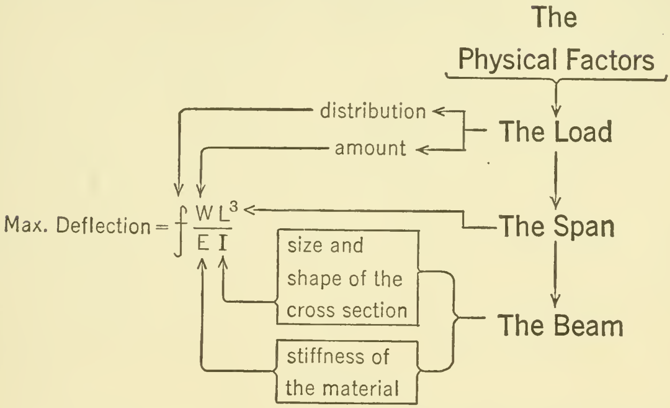
157. Fundamental Form for Maximum Deflection. In reviewing the curves in Table III, one cannot avoid noting that in cases where the loading is regular the value of the maximum deflection occurs as a fraction of $WL^3/(EI)$. This fact was forecast in § 149 from consideration of the physical factors entering the problem. Hence the following fundamental form for an expression giving maximum deflection in cases of regular loading is

$$y = f\frac{WL^3}{EI},$$

where f is a multiplicative constant. This expression can be

* As pointed out in the footnote, page 170, areas above the axis are regarded as positive and those below are regarded as negative.

related to the physical factors, load, span, and beam, by means of the following diagram (compare § 149).



158. Application of the Special Equations. A. Standard Cases. For the usual case of distributed and concentrated loading, the equations heretofore developed together with the cases listed in Table III furnish a simple method of determining deflections. A straight substitution of values in the equations will suffice. It may be worth while, however, to point out that when E is expressed in pounds per square inch and I in inches to the fourth power, the load must be expressed in *pounds* and the span in *inches*. This will give the deflection in inches, which is usual.

B. Simple Combinations. Slopes and deflections can be established by combination in the same manner as outlined for shears and moments in § 126. Thus when a loading can be analyzed into two or more standard cases, the values of the slope and deflection can be computed for each case and combined. If maximum values only are required and the point at which these values occur can be determined by inspection, direct computation and addition will be most satisfactory. If the entire curve is wanted or if the point of maximum deflection cannot be located by inspection, probably it will be best to draw the component curves at scale and add their ordinates together by use of dividers. When carefully executed on a large scale this

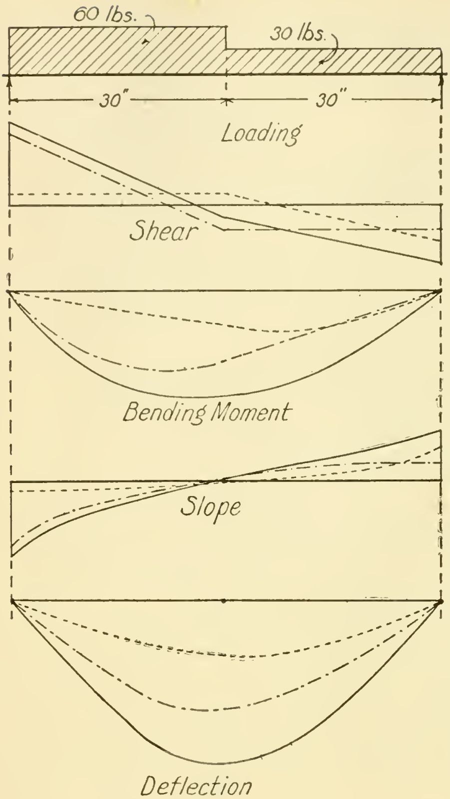


FIG. 270

method will give results that are sufficiently accurate for ordinary structural purposes.

C. NUMERICAL EXAMPLE. Let it be required to find the slope and deflection curves for the loading shown in Fig. 270. Such a case does not appear in Table III. But by breaking it down into two cases a result can readily be obtained. First consider the load of 30 lbs. distributed over the right-hand half of the beam. This case will correspond to No. 12 in the Table, provided that $(b - (d/2))$ is zero. Then the equation for deflection at any point becomes, by substitution of the values in this special case,

For the left half of the curve:

$$EIy = \frac{(30)(15)x}{(24)(60)}[4x^2 - 4(\overline{60}^2 - \overline{15}^2) + \overline{30}^2]$$

$$= \frac{5x^3}{4} - \frac{63{,}000x}{16}.$$

For the right half of the curve:

$$EIy = \frac{5x^3}{4} - \frac{63{,}000x}{16} - \frac{30}{(24)(30)}(x - 45 + 15)^4$$

$$= \frac{5x^3}{4} - \frac{63{,}000x}{16} - \frac{1}{24}(x - 30)^4.$$

From this equation the curve shown by the dotted line, Fig. 270, is established. By an exactly similar process, the broken line curve can be drawn for the 60 lb. load at the left. The total deflection is given by the solid line.

In general the equations set up for partial distributed loads merge into those for concentrated and for distributed loads as the partial load covers a lesser or greater part of the entire span.

D. Deflections by Approximation. Since the *allowable* deflection in any case (see E below) is largely a matter of individual judgment, great precision in the results of computations is not ordinarily demanded. Therefore, it is sometimes possible to arrive at satisfactory conclusions by well-considered approximations.

Thus, in the case worked out in C above, the maximum deflection is obviously greater than it would be if the load of the lesser intensity were continued all across the span. Also it is less than if the greater intensity were continued across. As a matter of fact an average between these two will come very close to the actual maximum deflection.

E. Allowable Deflections. As to what constitutes an allowable deflection, no rule can be stated. Everything depends on the particular case and that means on an individual judgment of the case. The only rule in general use seems to be that, for the sake of preventing the cracking of plastered ceilings, it is not desirable to allow deflections which exceed 1/360 of the span.

159. Other Methods for Determining Deflections. The determination of deflections by the methods given in the preceding articles is, in practice, subject to sharp limitations. When loading systems become complex, the elastic curve is a composite of several distinct curves, each having its own equation. There may be several real constants of integration to be dealt with. In such a case, the formulas become long and involved, running into the higher powers of x. (See cases 11, 12, 15, 17, Table III.) Again, when actual cases are dealt with, the numbers to be handled often run well into the millions, and the whole computation becomes very cumbersome and laborious.

For these reasons, other methods of finding deflections have been devised. Those methods are given in many standard texts. For the accurate solution of complex cases or of cases in which many computations are to be made, they will be found useful. However, the principles covered in the preceding articles are necessary to a proper understanding of such methods. For an occasional computation of a relatively simple case, the methods given above are satisfactory.

160. Summary. Before we leave the subject of deflections, it is worth while to summarize a few of the results obtained and the limitations imposed in this chapter.

A. Results.

(1) A complete set of relations between load, shear, moment, slope, and deflection.

(2) A set of values for maximum deflection which take the general form given in the diagram in § 157.

B. Limitations. The results obtained are limited by the following assumptions made in the derivations.

(1) That the unit stresses involved are less than the elastic limit. (§ 149.)

(2) That the curvature of the elastic curve is very slight. (§ 152.)

(3) That the material and cross section of the beam are constant throughout its length. (§ 153.)

Of these limitations, (3) is the only serious one in the design of structures. For cases in which the I is variable, special methods are worked out in more extended texts.

PROBLEMS

1. What is the maximum deflection of a timber beam, 8″ × 10″ × 15′ 0″, and which carries a uniformly distributed load of 4,000 lbs.?
2. What is the deflection of a 24″ × 100 lb. I beam 18′ 0″ long and which carries its full safe load concentrated at the center?
3. What will be the deflection of the beam in Problem 2 if the load is increased to 125,000 lbs.?
4. A bar of brass 1″ deep and 2″ wide spans between two supports 5′ 0″ apart and projects 1′ 8″ beyond each support. It carries a load of 50 lbs. on each end. What is the deflection upward at the middle and the radius of curvature?

5. A $6'' \times 12\frac{1}{4}$ lb. I beam rests on two supports which are 8′ 0″ apart and carries a load of 4,000 lbs. concentrated at its center. Draw curves showing slope and deflection throughout the span giving amounts at the ends and center of the beam.
6. Let the beam in Problem 4 have its loads replaced by a single load of 50 lbs. at the center of the span. What will be the upward deflection of the ends of the beam?
7. A wooden beam 4″ wide and 6″ deep spans 8′ 0″ and carries a uniformly distributed load which causes a deflection of $\frac{1}{4}''$. What is the maximum bending unit stress?
8. A piece of wood 1″ deep and 2″ wide spans 3′ 4″ and carries a uniformly distributed load of 50 lbs. Draw the load, shear, moment, slope, and elastic curves, and a curve showing how the radius of curvature varies from end to end.
9. A $7'' \times 15$ lb. I beam is used as a cantilever 5′ 6″ long. It carries a uniformly distributed load of 1,000 lbs. and a load concentrated at the end of 500 lbs. What is the deflection of the end?
10. In Problem 9 let the beam be extended to 3′ 0″ beyond the concentrated load. The extension carries no load. What is the deflection of the end?
11. If the beam in Fig. 251 is a $12'' \times 31\frac{1}{2}$ lb. I beam, what is the maximum deflection and where does it occur? (See Fig. 270, Graphic Solution suggested.)
12. What is the maximum deflection in a $2'' \times 6''$ simple wooden beam 10′ 0″ long when it is loaded to its full safe load in bending?
13. A wooden cantilever beam is 10′ 0″ long and $4'' \times 6''$ in section. A load of 100 lbs. at the end produces a deflection of $\frac{1}{2}''$. What is the modulus of elasticity of the wood?

CHAPTER XVIII

BENDING UNDER RESTRAINT

161. Introduction. The discussion of stresses and deflections in beams given in Chapters XIII to XVII assumes that the beam is free to deform naturally, as shown in Fig. 261, § 149. In such a case, the end planes of the beam are tilted from the vertical and the elastic curve, over the support, has a definite slope. (Footnote, page 211.)

In certain cases which arise in practice this condition does not exist, on account of the manner in which the beam is fastened to its supports. When a beam is so fastened at the ends that the end planes must remain in one position, regardless of this *tendency* to tilt, the ends are said to be ***fixed*** in position or ***restrained.***

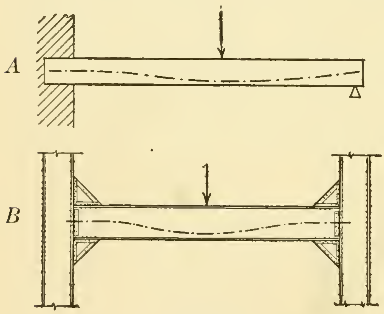


Fig. 271

Fixation or *restraint* can be accomplished in a number of ways. Two of these ways are shown in Fig. 271. In (A) one end of the beam is built into a wall, which holds the end of the beam in a fixed position. In (B) the beam is fastened to heavy columns which keep the ends of the beam vertical. In either case the important fact is that one or both of the planes forming the ends of the beam are *fixed* or *restrained* in a vertical position. Therefore the elastic curve of the beam must be horizontal at the restrained end or ends, as shown by the dot and dash lines, Fig. 271.

Another common case of restrained bending is shown in Fig. 272A. A single beam is shown resting on three supports. If these supports carried two simple beams, the beams would deform as shown in Fig. 272B. But if the material of which the beams are made is continuous over the support, the two end planes,

a and a', must have the same inclination at the support; i.e., the elastic curve must be a continuous curve as shown by the dotted line in Fig. 272A. In this case, each beam serves to *fix* or *restrain* the adjoining beam. A similar case arises whenever a single beam rests on any number of supports greater than two.

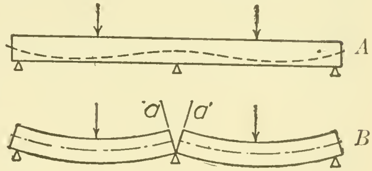


FIG. 272

In each of the cases outlined above, it is clear that the form of the elastic curve is quite different from that of an unrestrained beam and from that fact we may infer that the shears and bending moments are also of different amounts and are differently distributed. The purpose of this chapter is to develop the principles governing the determination of the shears, bending moments, and reactions in such cases. When these are known, the operations of investigation and design can be carried out just as in Chapters XIV and XV.

It is important to note that, in the two cases illustrated in Fig. 271, the elastic curve is always horizontal at the support unless the restraint imposed is incomplete. In practice, complete restraint seldom can be attained owing to the deformation of the supports themselves. This question will be considered further in § 167. For the purposes of the present discussion we shall use the word ***fixed*** to indicate complete restraint. In such a case, the end planes of the beam are vertical and the elastic curve is horizontal at the fixed end.

In the case shown in Fig. 272, the restraint imposed at the center support will be complete if the loads and spans are symmetrical; otherwise the end planes of the adjoining parts will have a common angle of inclination *other than vertical* and the elastic curve will have a definite slope over the support.

162. General Phenomena of Restrained Bending. In order to study the problems of restrained bending, let us take a beam free to deform naturally under a load, as shown in Fig. 273A. Now imagine that strong arms projecting upward are fastened to

the end planes. Next let equal and opposite horizontal forces be applied to the arms, as shown in *B*. These forces will gradually push the ends of the beam into a vertical position and will hold them there so long as the forces are maintained at a sufficient amount. We now have a case of bending under complete restraint.

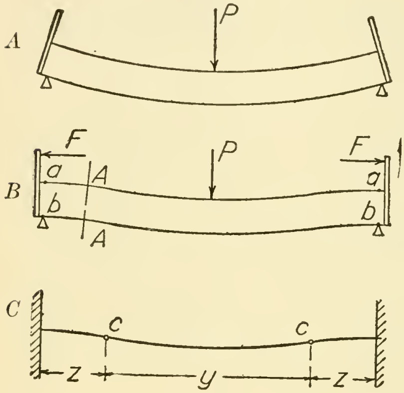


FIG. 273

It is evident that the arms must be pulling on the top of the beam (causing tensile stress at *a*), and pushing on the bottom (causing compressive stresses at *b*); in other words, they are producing a *negative* bending moment (§ 122) at the ends of the beam. The elastic curve of the beam must change to meet the new conditions. It must be horizontal over the support and convex upward throughout so much of the beam as is in the tension at the top surface. But near the center of the beam the load *P* causes the beam to take on a curvature which is convex downward. In such a case (see Fig. 273*C*) the restrained beam is analogous to two cantilever beams of length *z*, supporting on their ends a simple beam of length *y*.

Now let us take a case where restraint is applied to one end only, as in Fig. 274. As the plane of the left end of the beam is forced toward the vertical by the force *F*, the load *P* will be slightly raised; the reaction R_1 will be lessened and the reaction R_2 will be increased. This case is analogous to the cantilever supporting one end of a simple beam. Obviously a continuous beam (Fig. 272) is merely a collection of cases similar to the preceding, except that the restraint imposed is more or less complete, depending on the loadings and on the spans.

163. The Problem is Statically Indeterminate. In Fig. 274*A*, let us assume that the load and the span are known, and that the reactions are required. Take a center of moments at R_2. Evi-

dently the amount of R_1 depends on the moment of F about R_2. Thus the problem is seen to be statically indeterminate until the moment required to produce restraint is known. But, from the very definition of restraint, this moment is dependent on the elastic properties of the beam. Again, let us attempt to determine the bending moment at the section AA, Fig. 274A. It will be seen that this is impossible until the amounts of the reactions are known. These in turn depend on the moment of F. We can now draw the following conclusions concerning the determination of shears, bending moments, and reactions in cases of restrained bending.

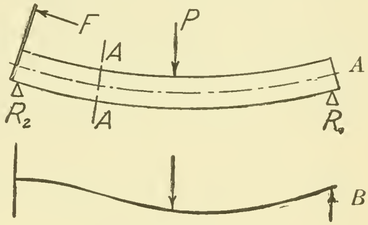


FIG. 274

(1) The problem is statically indeterminate.

(2) Its solution depends on determining the moment required to produce restraint.

(3) This solution must take account of the elastic properties of the beam and particularly it must take account of the shape of the elastic curve.

164. Limitations. As pointed out above, the stresses and reactions which accompany restrained bending depend on the elastic properties of the beam as well as on the applied forces; therefore the solutions are necessarily more complex than those for simple beams. In order to keep them as simple as possible, the following assumptions are made.

(1) That in all cases the supports of the beams are on the same level. (See also § 174.)

(2) That by *fixation* we mean the application of forces that will maintain the end plane or planes of the beam in a vertical position and, as a consequence, will force the elastic curve to be horizontal over the support or supports.

(3) That the general theories of bending stresses and deflections, as developed in Chapters XIV and XVII, apply to this

case, and that therefore the limitations on those theories (§§ 132 and 160) also apply.

(4) That the material and the cross section of the beam are constant throughout its length (§§ 153 and 160).

In more extended texts, solutions are worked out for cases where the supports are at different levels, and also for cases of beams of variable cross section.

165. Beams Fixed at One End. A. The General Principle. Let the beam shown in Fig. 275 be fixed at the left end and simply supported at the right end. If the support R be removed, the beam becomes an ordinary cantilever beam, and the deflection at the right end, due to the load P, can be determined by methods like those used in Chapter XVII. (See also Table III.) Next let the load P be removed and some (unknown) force R be applied (upward) at the right end. This will cause

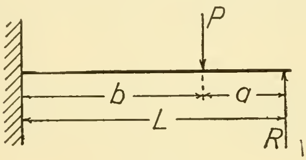


Fig. 275

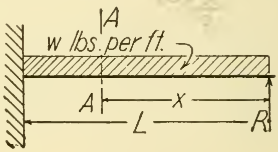


Fig. 276

an upward deflection which can be expressed in terms of the unknown force R (§ 154). In the beam as shown these two forces are acting and the resultant deflection of the right end is zero. This principle forms the basis of the following solutions.

B. Load Uniformly Distributed. Let the beam be as shown in Fig. 276. Treating this beam as a cantilever of length L (see § 155), the deflection of the right end is

$$(1) \qquad -y = \frac{WL^3}{8EI}.$$

The deflection (upward), due to R (§ 154), is

$$y = -\frac{RL^3}{3EI}.$$

Therefore, since the actual deflection is zero,

$$\frac{WL^3}{8EI} - \frac{RL^3}{3EI} = 0, \qquad R = \frac{3W}{8}.$$

We can now proceed to determine the shear diagram in the usual manner, working from the right. Thus, at the section AA, distant x from the right end, the shear is

$$V = \tfrac{3}{8}wL - wx.$$

In this equation $V = 0$ when $x = (3/8)L$, and when $x = L$, $V = -(5/8)wL$, or $-(5/8)W$. Similarly, the moment at the section AA is

$$M = \frac{3wLx}{8} - \frac{wx^2}{2}.$$

From this equation we find that the maximum positive moment, which occurs when $x = (3/8)L$, is $9wL^2/128$; or $9WL/128$; the maximum negative moment (when $x = L$) is $-wL^2/8$ or $-WL/8$, and that the moment is zero when $x = 3L/4$, as well as when $x = 0$. The slope and deflection diagrams for this case are given in Table III. They can be worked out in a manner similar to that of § 154.

C. Concentrated Load. Let the beam be as shown in Fig. 275. The general procedure is the same as in B, above. The deflection of the right end, treating the beam as a cantilever (see Table III), is

$$-y = \frac{Pb^2(2b + 3a)}{6EI} = \frac{Pb^2(3L - b)}{6EI}.$$

The (upward) deflection due to R is

$$y = -\frac{RL^3}{3EI}.$$

Therefore

$$\frac{RL^3}{3EI} = \frac{Pb^2(3L - b)}{6EI}, \qquad \text{or} \qquad R = \frac{P}{2L^3}b^2(3L - b).$$

The shear at the fixed end is

$$V = P - R = \frac{P}{2L^3}(3aL^2 - a^3).$$

The bending moment at any point distant x from the right support (when $x > a$) is

$$M = Rx - P(x - a).$$

The bending moment at the restrained end then becomes

$$\begin{aligned} M' &= RL - P(L - a) \\ &= \frac{P}{2L^2}[3L(L - a)^2 - (L - a)^3] - P(L - a) \\ &= -\frac{P}{2L^2}a(L^2 - a^2). \end{aligned}$$

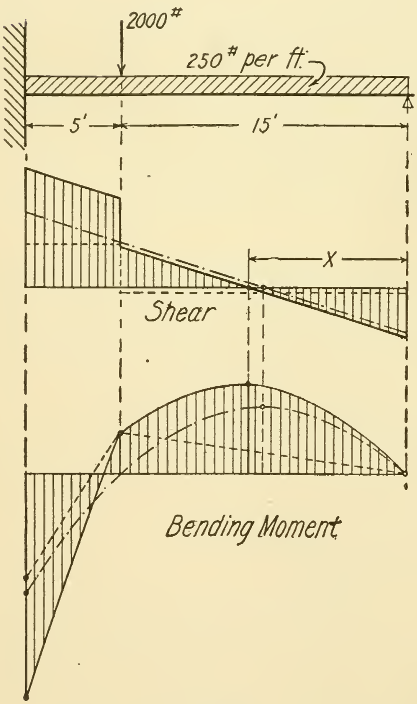


Fig. 277

The position of the load which will cause the greatest moment at the restrained end can now be determined. Let a be considered as variable, then

$$\frac{dM'}{da} = \frac{3Pa^2}{2L^2} - \frac{P}{2}.$$

Now putting this equal to zero and solving for a, we find

$$a = 0.577L.$$

Similarly, it can be shown that the position for maximum positive moment is $a = 0.375L$. The slope and deflection diagrams for this case are given in Table III. They can be derived in a manner similar to that of § 154.

D. Combined Loading. When a restrained beam carries a loading which is partly distributed and partly concentrated, as in Fig. 277, a general solution, while possible, becomes very complex, and hardly worth while. It is much simpler to treat each load separately, as in B and C above, making a separate shear diagram and a bending-moment diagram for each. The combined effect of the loads then can be found by combining these diagrams as in § 158, and as shown in Fig. 277.

Another way of solving this case is to find, separately, the reaction at the right due to the uniform load, and that due to the concentrated load, as in B and C above. Then combine these into one and, by the free-body method, compute the shears and bending moments, starting at the free end of the beam.

PROBLEMS

✓ 1. A beam 12′ 0″ long is fixed at one end and supported at the other. It carries a load of 1,000 lbs. at the center of the span. Determine the maximum shear, the maximum positive and negative bending moments, and the point of contraflexure.

2. Determine the amounts and positions of the maximum bending moments and the maximum shear in a beam 12′ 0″ long, fixed at one end, and carrying a load of 1,000 lbs. per ft. as well as a concentrated load of 5,000 lbs. at 4′ 0″ from the fixed end.

✓ 3. Determine the amounts and positions of the maximum shear and maximum bending moment in a beam 12′ 0″ long and fixed at one end. There is a uniformly distributed load of 1,000 lbs. per foot extending from the fixed end to within 5′ 0″ of the other end. There is also a load of 500 lbs. per ft. on the rest of the beam. (Refer to Table III, cases 3 and 5.)

4. Draw the shear, bending moment, slope and deflection diagrams for a beam fixed at one end, carrying a concentrated load of 1,000 lbs. at the center of a span of 12′ 0″. Let the slope and deflection diagrams be given in terms of EIv and EIy.

5. Draw the same four diagrams as in Problem 4 for a beam 12′ 0″ long and loaded with a distributed load which varies from 100 lbs. per ft. at the fixed end to zero at the supported end.

✓ 6. Determine the position and amount of the maximum deflection in Problem 1. Let the beam be of wood, 3″ × 6″ in cross section.

166. Beams Fixed at Both Ends. A. The General Principle. In Fig. 273*B*, let the part of the beam to the left of section AA be cut loose and shown as a free body in Fig. 278. The reaction R is the same as would occur in a simple beam since the

forces F have no vertical components. Let o be taken as a center of moments. Then evidently the bending moment on the section AA is $Rx - Fm$. But Rx is the same moment that would occur in a simple beam, and Fm is the moment of restraint. Thus the resultant moment is merely the moment that would exist on a simple beam minus the moment due to restraint. Or, if we let M represent the moment due to any loading on a simple beam, and M' represent the moment due to restraint, the resultant moment, in general, will be $M - M'$.

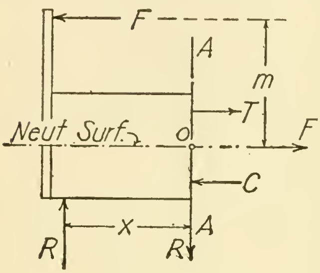


Fig. 278

Since it is the resultant moment that produces the deformations of elementary slices through the beam (§ 150), it is evident that the slope and the deflection of the elastic curve will be those due to a moment of $M - M'$. In the case of a simple beam with a given loading, the slope and the deflection due to the moment M can be determined by the methods given in Chapters XIII and XVII. Also the slope and deflection due to any unknown moment can be expressed *in terms of that moment* (§§ 152–160). By adding the two expressions thus found, we can get an expression (involving the unknown moment of restraint) for the total slope or deflection throughout a restrained beam. But the slope or deflection (or both) of such a beam will have definitely fixed values at certain points. Thus, in Fig. 273B, the slope of the elastic curve over the support is zero. These facts enable us to write an equation between the known slope or deflection and its value expressed in terms of the unknown moment of restraint. When this equation is solved, the moment of restraint is known and a complete solution for the shear, moment, slope, and deflection is possible.

B. Uniformly Distributed Load. In the following solution, the moment and slope which occur in a simple beam with similar loading are denoted by the usual characters, M and v. The

corresponding quantities which would result from the restraining moment alone are indicated by the same characters primed. The final resultant moment and slope (obtained by adding the two above) are indicated by a double prime.

In Fig. 279*A* is shown a beam fixed at both ends and uniformly loaded. Let it be cut free by planes passed close to the supports. The free body is shown in Fig. 279*B*. At each end there exists on the cut section a shear of $W/2$, and an unknown restraining moment which is indicated as M'.

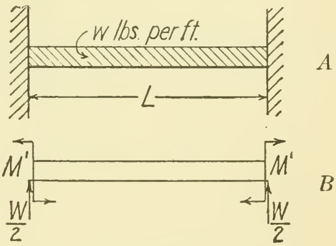


Fig. 279

The slope of the elastic curve at the left end due to the given loading acting on a *simple* beam (Table III) would be

$$v = \frac{WL^2}{24EI}.$$

The slope at the same point due to the restraining moment M' acting alone (Table III, or § 155C) would be

$$v' = \frac{M'L}{2EI}.$$

Now the actual slope of the curve at the end is zero; therefore

$$\frac{WL^2}{24EI} + \frac{M'L}{2EI} = v'' = 0,$$

$$(2) \qquad M' = -\frac{WL}{12} \quad \text{or} \quad -\frac{wL^2}{12}.$$

Now taking a section at a distance x from the left support, we find that the bending moment is

$$(3) \qquad M'' = -\frac{WL}{12} + \frac{W}{2}x - \frac{wx^2}{2L}.$$

By substituting $L/2$ for x in the above equation we find the value of M'' at the center of the beam to be $wL^2/24$ or $WL/24$.

Similarly by putting $M'' = 0$ we find that the inflection point is at a distance $0.2114L$ from the left end.

The slopes and deflections can be determined in a similar manner by adding the effect of the load acting on a simple beam to that of the restraining moment whose value is now known. The values are given in Table III.

C. Concentrated Load. In this case the elastic curve is made up of two separate curves: one for the part to the left of the load, the other for the part to the right of the load. These curves have a common slope and a common deflection where they meet, under the load. Moreover, the slopes and deflections are zero at either end of the beam. These conditions are the basis for the solution.

In Fig. 280A the loading diagram is shown. In Fig. 280B the beam is shown as a free body cut by planes passed near the supports. The unknown shears and moments (V'', V''', M'', M''') are indicated by arrows. The moment, slope, and deflection *throughout* the left-hand part of the beam will be indicated by the usual symbols M, v, and y; those for the right-hand part by M', v', and y'.

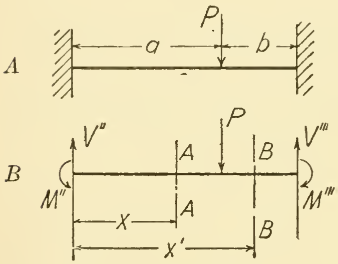


Fig. 280

For the left-hand part of the beam, let a free body be cut by the plane AA, at a distance x from the left support; then

(4) $$M = M'' + V''x.$$

Substituting and integ ating (§ 153), we find

(5) $$EI\,' = M''x + \frac{V''x^2}{2} + C.$$

But $v = 0$ when $x = 0$. Therefore $C = 0$, and

(6) $$EIv = M''x + \frac{V''x^2}{2}\cdot$$

Integrating again (§ 153), we get

$$EIy = \frac{M''x^2}{2} + \frac{V''x^3}{6} + C_1 \cdot$$

But $y = 0$ when $x = 0$. Therefore $C_1 = 0$, and

$$EIy = \frac{M''x^2}{2} + \frac{V''x^3}{6} \cdot \tag{7}$$

For the right-hand part of the beam, take a section at BB. The bending moment, figured from the left, is

$$M' = M'' + V''x' - P(x' - a). \tag{8}$$

Substituting and integrating, we find

$$EIv' = M''x' + V''\frac{x'^2}{2} - \frac{P}{2}(x' - a)^2 + C_2.$$

But the slopes v and v' are equal when $x = x' = a$. Therefore $C_2 = 0$, and

$$EIv' = M''x' + V''\frac{x'^2}{2} - \frac{P}{2}(x' - a)^2. \tag{9}$$

Integrating again, we find

$$EIy' = \frac{M''x'^2}{2} + \frac{V''x'^3}{6} - \frac{P}{6}(x' - a)^3 + C_3.$$

But the deflections y and y' are equal when $x = x' = a$. Therefore $C_3 = 0$, and

$$EIy' = \frac{M''x'^2}{2} + \frac{V''x'^3}{6} - \frac{P}{6}(x' - a)^3. \tag{10}$$

Now $v' = 0$ when $x' = L$; also $L - a = b$. Therefore, from (9),

$$0 = M''L + \frac{V''L^2}{2} - \frac{P}{2}b^2,$$

$$V'' = -\frac{2M''}{L} + \frac{P}{L^2}b^2. \tag{11}$$

Also $y' = 0$ when $x' = L$, and $L - a = b$. Therefore, from (10),

$$0 = \frac{M''L^2}{2} + \frac{V''L^3}{6} - \frac{P}{6}b^3. \tag{12}$$

Substituting the value of V'' from (11) in (12), we obtain

$$0 = \frac{M''L^2}{2} - \frac{M''L^2}{3} + \frac{PLb^2}{6} - \frac{P}{6}b^3,$$

$$-\frac{M''L^2}{6} = \frac{P}{6}(Lb^2 - b^3),$$

$$M'' = -\frac{Pb^2}{L^2}(L - b) = -\frac{Pab^2}{L^2}. \tag{13}$$

Substituting this value in equation (11), we have

$$V'' = \frac{Pb^2}{L^3}(b + 3a). \tag{14}$$

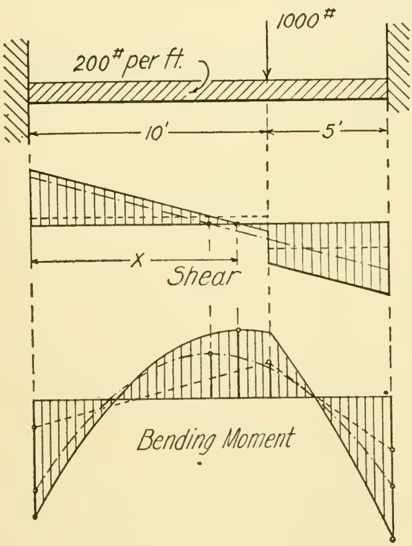


Fig. 281

With the values of M'' and V'' thus found, the values for the moment at any point on the beam can be found by substitution in equation (4) or (8). The slope can be found from (6) or (9) and the deflection from (7) or (10). The principal values are given in Table III.

D. Combined Load. As pointed out in § 165D, a general solution for this case is hardly worth while. A specific case is shown in Fig. 281. Let it be required to determine the shear and bending moment diagrams. The shear at each end due to the uniform load, as given on page 235, is

$$\frac{200 \times 15}{2} = 1{,}500 \quad \text{lbs.}$$

The shear at the left end due to the concentrated load is determined from equation (14) above, and is, approximately,

$$\frac{1{,}000 \times 25}{3{,}375}(5 + 30) = 260 \qquad \text{lbs.}$$

Thus the total shear at the left end is 1,760 lbs. The bending moment at the left end due to the uniformly distributed load alone, as given in equation (2) on page 235, is

$$\frac{-\,200 \times 15 \times 15}{12} = -\,3{,}750 \qquad \text{lbs. ft.}$$

The bending moment at the same section due to the concentrated load, as given in equation (13) on page 238, is

$$\frac{-\,1{,}000 \times 10 \times 25}{225} = -\,1{,}111 \qquad \text{lbs. ft.}$$

Thus the total moment at the left end is − 4,861 lbs. ft. The bending moment at a point distant 3′ from the left end is

$$(1{,}760 \times 3) - (4{,}861) - (200 \times 3 \times 1\tfrac{1}{2}) = -\,481 \text{ lbs. ft.}$$

The complete diagrams are given in Fig. 281.

167. Effects of Restraint. A comparison between the bending moments for beams with fixed ends and those for simple beams with the corresponding spans and loadings (see Table III) will show that the maximum bending moment for a beam with fixed ends is, in general, less than that for a simple beam. Deflections follow the same rule. This is equivalent to saying that the effect of restraint is to increase both the strength and the stiffness of a given beam. Or again, a beam which is restrained may be made smaller than one which is not restrained. It should be noted however that this increase in strength and stiffness, as determined in §§ 165 and 166, does not occur unless the supports do actually furnish an absolute restraint. Except under very favorable circumstances, this condition does not exist, since the deformations of the supports are usually sufficient to allow some motion of the ends of the beam.

While it is true that conditions of absolute restraint are rare, the values already worked out are not worthless, for the true bending moments and deflections of any partly restrained beam will be found somewhere between those for a beam with fixed ends, as herein determined, and those for a simple beam.

PROBLEMS

1. A beam 12′ 0″ long is fixed at both ends. It carries a load of 1,000 lbs. at the center of the span. Determine the maximum shear, maximum positive and negative moments, and points of contraflexure.
2. A beam 12′ 0″ long is fixed at both ends. It carries a uniform load of 1,000 lbs. per foot and a concentrated load of 5,000 lbs. at 4′ 0″ from the left end. Find the maximum shear and the maximum positive and negative moments; also the points of contraflexure.
3. Draw the shear, moment, slope and deflection diagrams for a beam 12′ 0″ long, fixed at both ends and carrying a uniform load of 1,000 lbs. per foot. Derive the equations of each curve and the amounts of all critical values. Let the slope and deflection be given in terms of EIv and EIy.
4. Derive the shear, moment, slope, and deflection diagrams for a beam fixed at both ends and carrying a load which varies from w lbs. per foot at the left end to zero at the right.
5. If the beam in Problem 1 is a wooden beam, 2″ × 10″ in section, determine the maximum deflection.
6. If the beam in Problem 2 is a 10″ × 25 lb. I beam, determine the amount and position of the maximum deflection.

168. Continuous Beams. It has been shown already (§ 161) that when a single beam rests on more than two supports, the continuity of the beam over the interior supports constitutes a restraint which is of the same general nature as in the cases of restrained bending worked out in §§ 162–167.

In reinforced concrete work the continuity of beams over their supports is a natural outgrowth of the way in which the work is done best. In other materials, the continuity of beams is not so general, but it is by no means rare.

The added strength and stiffness that result from the restraint imposed by continuity (§ 167) are factors making for economy in design.

When a series of separate beams rests on a series of supports as shown in Fig. 282, the loads cause deflection and the elastic

curve of each beam takes the form characteristic of the loading on that beam. But if the same loads are carried on a single continuous beam, the elastic curve is forced to take on a different character, and corresponding changes take place in the distribution of bending moments and shears (Fig. 283). Imagine the two adjacent non-continuous beams in Fig. 284 to be deflected by the applied loads as shown. Now imagine strong arms attached to the ends as shown at a and b. Next let the forces X and Y be applied to the arms, forcing them together. This operation will put tension on the tops of the beams where the arms are fastened and compression on the bottoms, i.e., it will set up a negative bending moment in the ends of the beams. The beams then will take on the form shown by the dotted lines. This, in effect, is what happens in every beam that is continuous over more than two supports. Continuity over supports is merely a special case of restrained bending. The

FIG. 282

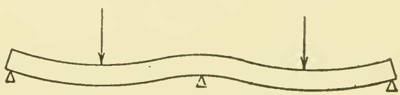


FIG. 283

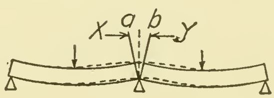


FIG. 284

analogy to a series of cantilever and simple beams which was pointed out in § 162 applies also to continuous beams. A continuous beam may have any number of supports greater than two.

169. General Principles for the Solutions. The case of a continuous beam is a statically indeterminate one, since there are at least three reactions (§ 40). However, the fact of the continuity of the beam over the supports furnishes the basis of a solution depending on the elastic qualities of the beam; for, if the beam is continuous, so is also its elastic curve. This fact supplies the needed equations, as shown in § 170.

Let Fig. 285 represent two adjoining spans cut from a continuous beam of any larger number of spans. In this solution, it is necessary to use two adjoining spans since we propose to use the fact of continuity over the support between them in the solution. These spans are under the action of the loading and the three reactions, and also the shears, V_1–V_3, and the moments, M_1–M_3, which exist on the cut sections at either end. The effects of all loads which may be on the spans to the left of R_1 will appear in the moment M_1 and in the shear V_1; and similarly at the right of R_3.

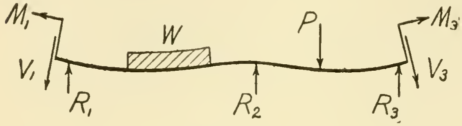


Fig. 285

If the cut section at the left is taken *close* to the support, any variation in V_1 (caused by a variation in the loading on the spans to the left) will cause a change in the reaction R_1 but will not otherwise affect the shears, moments, or deflections throughout the two spans. On the other hand, any variation in M_1 will affect the moments and deflections in both spans as well as the amount of the reaction R_2.* Similarly, at the right, the effect of all the loading to the right of R_3 will be transmitted over the support by the moment M_3. It is thus evident that in our solution (1) we must deal with at least two adjacent spans in order to introduce the element of continuity, (2) not more than two spans will be necessary since the effect of all other loadings will be found in the moments over the supports, (3) the moments over the supports must hold the key to the final solution.

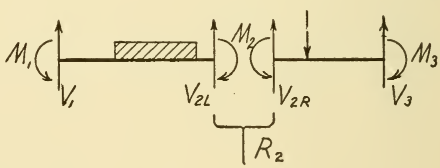


Fig. 286

* If this is not clear at first, imagine the moment M_1 to be largely increased. Because of the stiffness of the beam, this will tend to raise the load W and lighten the reaction R_2. Also the curvature of the beam over R_2 will be changed, indicating a change in the moment over the support which will then affect the moment and curvature throughout the second span.

The above considerations will become more evident if both spans are cut loose *close to the reactions,* and the unknown shears and moments are indicated as shown in Fig. 286. Since there is no appreciable distance cut out over the center support, the unknown moments on the two spans are equal and are so indicated. There are then seven unknown quantities, four shears and three moments. We have at our disposal the two conditions of equilibrium, $\Sigma V = 0$ and $\Sigma M = 0$. These applied to both spans will yield four equations. The continuity of the elastic curve over the support R_2 yields another, leaving two more to be found. This is all that can be accomplished with the two spans and it results in giving one equation with three unknown quantities. These three quantities usually may be chosen so that they are the three unknown moments. But, though they remain unknown at this stage, the equation establishes the *relation between them;* and this relation holds for *any* two spans in a beam, continuous over *any* number of supports.

Now, assuming that we have such an equation, turn to Fig. 287 and let us imagine that we apply it to the first and second spans at the left; then to the second and third and so on. It is evident that in a beam of n spans, this process will yield $n - 2$ equations. The two missing equations usually are supplied by known conditions concerning the moment over the end supports, as explained in § 170. When these equations are solved, all of the unknown moments over the supports will become known. Then, taking each span as a free body, we can determine the reactions and the shears.

Fig. 287

The equation of relation between the moments over any three adjoining supports is known as *the three-moment equation.* It evidently will be different for each different type of loading or span relation. In the following article the three-moment equation is developed for two typical cases.

170. Evaluations for Uniform Loading. A. Derivation of Three-moment Equation. Let Fig. 288 represent two adjacent spans of a continuous beam, taken as a free body, the notation

to be used being shown on the drawing. Let the unknown moments over the supports be M_1, M_2, and M_3. Now let a portion of the right-hand span be cut loose, and shown as free

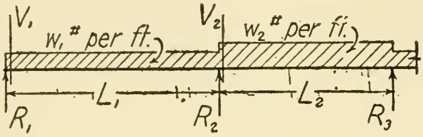


Fig. 288

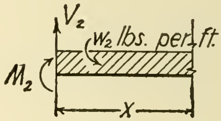


Fig. 289

body in Fig. 289. The unknown shear and moment at the left are indicated by M_2 and V_2; and these, together with the distributed load of w_2 pounds per foot, constitute the loading of the free body.

The moment on the right-hand section, which is at a distance x ft. from the left end, is

$$(15) \qquad M_x = M_2 + V_2x - \frac{w_2x^2}{2}.$$

Let x be taken equal to L_2. Then M_x becomes the moment at the right support, or M_3. Then, from equation (15), we have

$$(16) \qquad M_3 = M_2 + V_2L_2 - \frac{w_2L_2^2}{2},$$

$$(17) \qquad V_2 = \frac{M_3 - M_2}{L_2} + \frac{w_2L_2}{2}.$$

Turning to a consideration of the slope of the elastic curve in the beam, we get, by § 153, for the span shown in Fig. 289,

$$EIv = \int M_x dx.$$

Substituting the value of M_x from equation (15), and integrating, we find

$$(18) \qquad EIv = M_2x + V_2\frac{x^2}{2} - \frac{w_2x^3}{6} + C.$$

Then, by § 153, we have

$$(19) \quad EIy = \int EIvdx = M_2\frac{x^2}{2} + V_2\frac{x^3}{6} - \frac{w_2x^4}{24} + Cx + C'.$$

But we know that the deflection y is zero when x is zero. Therefore $C' = 0$. Also $y = 0$ when $x = L_2$. From this we can show by use of equation (19) that

$$C = -M_2\frac{L_2}{2} - V_2\frac{L_2^2}{6} + \frac{w_2L_2^3}{24}.$$

Substituting this value of C in (18), we get

$$(20)\quad EIv = M_2x + V_2\frac{x^2}{2} - \frac{w_2x^3}{6} - \frac{M_2L_2}{2} - \frac{V_2L_2^2}{6} + \frac{w_2L_2^3}{24}.$$

Taking the value of V_2 as given in equation (17), and substituting it in equation (20), we find

$$(21)\quad \begin{aligned} EIv = M_2\left(x - \frac{x^2}{2L_2} - \frac{L_2}{3}\right) + M_3\left(\frac{x^2}{2L_2} - \frac{L_2}{6}\right) \\ + w_2\left(\frac{L_2x^2}{4} - \frac{x^3}{6} - \frac{L_2^3}{24}\right), \end{aligned}$$

which gives the value of the slope of the elastic curve in the right-hand span in terms of the unknown moments. Obviously an equation for the slope of the elastic curve in the *left-hand* span will be the same except for a change in the subscripts on the values for moment, span, and loading. Thus, for the left span, letting x be measured from the *left* support as before, we find

$$(22)\quad \begin{aligned} EIv' = M_1\left(x - \frac{x^2}{2L_1} - \frac{L_1}{3}\right) + M_2\left(\frac{x^2}{2L_1} - \frac{L_1}{6}\right) \\ + w_1\left(\frac{L_1x^2}{4} - \frac{x^3}{6} - \frac{L_1^3}{24}\right), \end{aligned}$$

which is the slope of the elastic curve in the *left* span.

Now the elastic curve of the beam over the center support is *continuous*. Therefore the slope values given in equations (21) and (22) are the same at the point over the support, i.e., when $x = L_1$ in equation (22) and when $x = 0$ in equation (21). Making these substitutions for x and equating the results, we find

$$M_2\left(\frac{-L_2}{3}\right) + M_3\left(\frac{-L_2}{6}\right) - w_2\left(\frac{L_2^3}{24}\right)$$
$$= M_1\left(\frac{L_1}{6}\right) + M_2\left(\frac{L_1}{3}\right) + w_1\left(\frac{L_1^3}{24}\right),$$

or

(23) $$M_1L_1 + 2M_2(L_1 + L_2) + M_3L_2 = -\frac{w_1L_1^3}{4} - \frac{w_2L_2^3}{4},$$

which is the equation of relation between the three unknown moments over the supports (the three-moment equation), for the case of uniform loading.

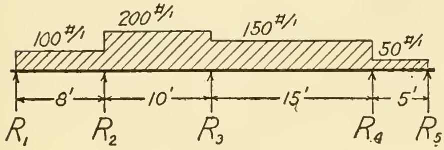


Fig. 290

B. A Numerical Application. Let the continuous beam with distributed loadings as shown in Fig. 290 be given and let it be required to draw the shear and bending moment diagrams. The moments over the supports will be indicated by M_1 to M_5. The shears at the right and left of any support will be indicated by V_{2R}, V_{2L}, V_{3R}, V_{3L}, etc.

If the beam is merely supported (not restrained) at the two outermost supports, the moments over these supports, M_1 and M_5, will be zero. Applying the three-moment equation derived in the previous paragraph to the first and second spans on the left of the beam, we get

$$0 + 2M_2(18) + M_3(10) = -\frac{100(8)^3}{4} - \frac{200(10)^3}{4},$$

or

(24) $$18M_2 + 5M_3 = -31{,}400 \quad \text{lbs. ft.}$$

Applying the same equation to the two middle spans, we find

$$M_2(10) + 2M_3(25) + M_4(15) = -\frac{200(10)^3}{4} - \frac{150(15)^3}{4},$$

or

(25) $$2M_2 + 10M_3 + 3M_4 = -35{,}312 \quad \text{lbs. ft.}$$

Applying it to the two spans on the right, we get

$$M_3(15) + 2M_4(20) + 0 = -\frac{150(15)^3}{4} - \frac{50(5)^3}{4},$$

or

(26) $$3M_3 + 8M_4 = -25,625 \quad \text{lbs. ft.}$$

Eliminating M_4 between equations (25) and (26), we have

(27) $$16M_2 + 71M_3 = -205,625 \quad \text{lbs. ft.}$$

Then, eliminating M_2 between equations (24) and (27), we get

$$M_3 = -2,674 \quad \text{lbs. ft.}$$

and, substituting this value in equation (24),

$$M_2 = -1,002 \quad \text{lbs. ft.}$$

and finally, from equation (26),

$$M_4 = -2,200 \quad \text{lbs. ft.}$$

Now let the left-hand span be taken as a free body, the cutting

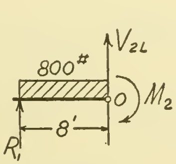


Fig. 291

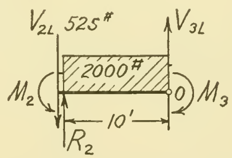


Fig. 292

plane being passed just to the left of R_2, as shown in Fig. 291. Taking a center of moments at o, we may write

$$M_2 = -1,002 \text{ lbs. ft.} = 8R_1 - 4(800), \quad \text{or} \quad R_1 = 275 \text{ lbs.}$$

Then, from the equation $\Sigma V = 0$, we find

$$V_{2L} = 800 - 275 = 525.$$

Taking the second span as a free body (Fig. 292), we find

$$M_3 = -2,674 \text{ lbs. ft.} = -10(525) - 2,000(5) - 1,002 + 10R_2,$$
$$R_2 = 1,358 \text{ lbs.}$$

Then since $V_{2L} + V_{2R} = R_2$, $V_{2R} = 833$. Continuing this process through the two remaining spans, we get

$$\left.\begin{aligned} V_{3L} &= 1{,}167 \\ V_{3R} &= 1{,}156 \end{aligned}\right\} R_3 = 2{,}323,$$

$$\left.\begin{aligned} V_{4L} &= 1{,}094 \\ V_{4R} &= 565 \end{aligned}\right\} R_4 = 1{,}659,$$

$$R_5 = -315.$$

The complete shear diagram and bending-moment diagram for this case are given in Fig. 293. Let the student check these

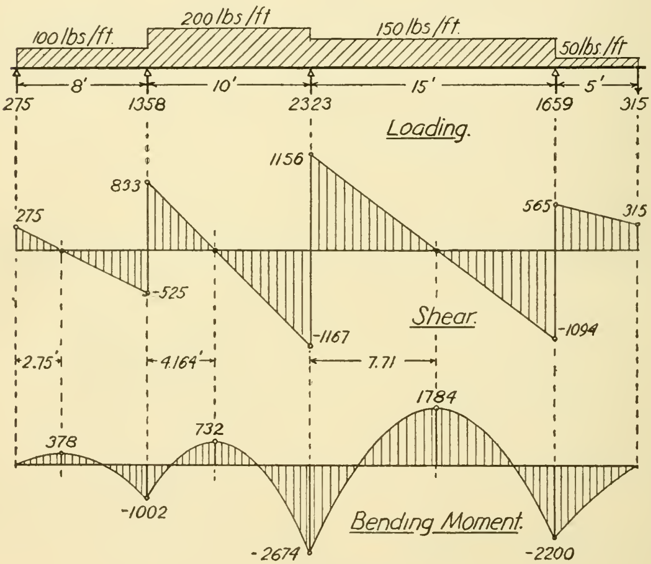


Fig. 293

diagrams with respect to the points of zero shear and the amounts of the maximum positive bending moments.

PROBLEMS

1. Draw the shear and bending moment diagrams for a continuous beam which rests on three supports (each span $= L$) and which carries a uniformly distributed load. Show the amounts of the reactions, shears, and moments and locate the points of contraflexure. (Compare with the case of a beam fixed at one end.)

2. Work out Problem 1 for the case of four supports.
3. Determine the shears, bending moments, reactions, and points of contraflexure throughout the continuous beam shown in Fig. 294.

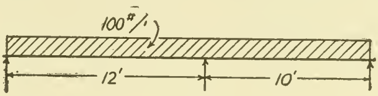


FIG. 294

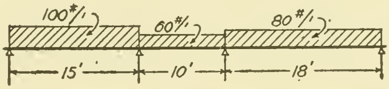


FIG. 295

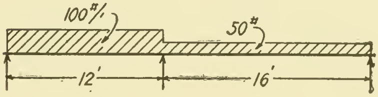


FIG. 296

4. Determine the same quantities as in Problem 3, for the case shown in Fig. 295.
5. If the loads in Problem 4 are to be carried on steel beams, will it require less metal if the beam is continuous or if several separate beams are used?
6. Determine the same quantities as in Problem 3, for the case shown in Fig. 296.

171. Concentrated Loadings. Let Fig. 297 represent two adjacent spans in a continuous beam with concentrated loadings. Let the unknown shears and moments be denoted as in the previous article.

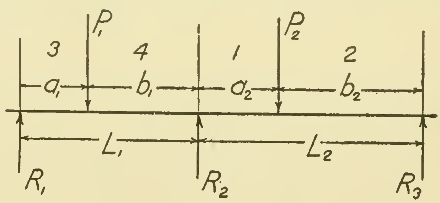


FIG. 297

The derivation of the three-moment equation is similar to that in § 170, but more involved. Equations for the shear, moment, slope, and deflection are written for each part of each span. The connecting principle is that the slope in span 2 is equal to that in span 1 when both equations are made to refer to a point over R_2. The resulting equation of relation for this case is

$$M_1L_1 + 2M_2(L_1 + L_2) + M_3L_2$$
$$= -\frac{P_2b_2}{L_2}(L_2^2 - b_2^2) - \frac{P_1a_1}{L_1}(L_1^2 - a_1^2).$$

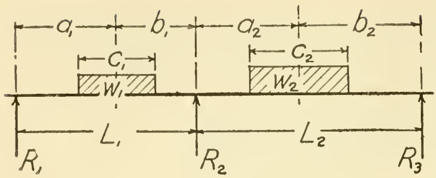


Fig. 298

This equation may be made to apply to the case of several loads in each span by simple summation of the effects of the loads, taken separately.

172. Partial Distributed Loads. For the case shown in Fig. 298, the equation of relation is obtained in a similar manner to the two cases above. It is

$$M_1L_1 + 2M_2(L_1 + L_2) + M_3L_2 = -\frac{w_1c_1a_1}{4L_1}[4(L_1^2 - a_1^2) - c_1^2]$$
$$-\frac{w_2c_2b_2}{4L_2}[4(L_2^2 - b_2^2) - c_2^2],$$

in which w_1 and w_2 represent the load per linear unit.

173. Equal Spans and Loadings. When the spans of a continuous beam are all equal and each span is uniformly loaded,

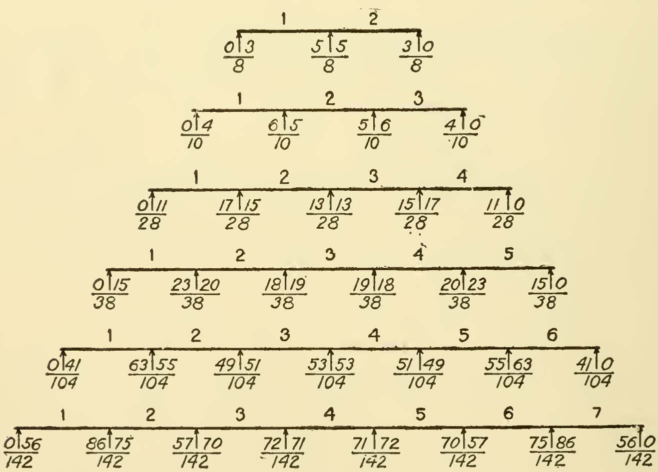


Fig. 299. Shears in Continuous Beams.

Equal spans (L) and uniform loading (w lbs. per ft.). Figures given are coefficients of wL; e.g. for a beam of four spans, shears at left and right of first span are $\frac{11}{28}wL$ and $\frac{17}{28}wL$ respectively; for second span, $\frac{15}{28}wL$ and $\frac{13}{28}wL$, etc.

the shears and the bending moments can be conveniently tabulated for reference. Such a table is given in Figs. 299 and 300.

Uniform loading is the usual condition in structural design.

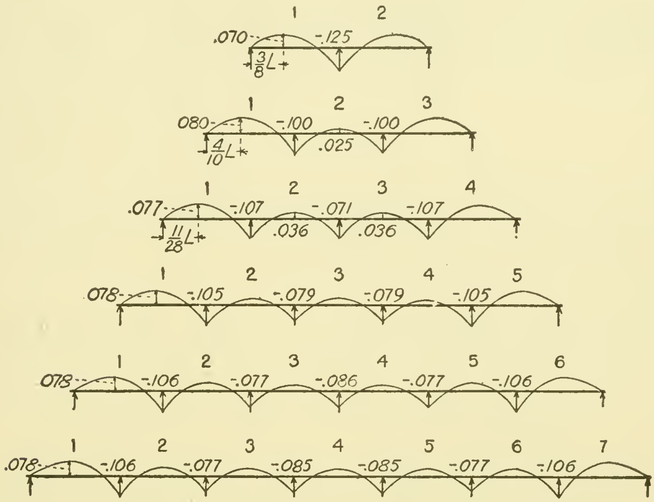


FIG. 300. Bending Moments in Continuous Beams.

Equal spans (L) and uniform loading (w lbs. per ft.). Figures given are coefficients of wL^2; e.g. in a beam of four spans, maximum positive moments are $.077wL^2$ and $.036wL^2$; maximum negative moments are $-.107wL^2$ and $-.071wL^2$.

Where concentrated loadings occur, their effects can be computed by the use of § 171, or by one of the means of approximations given in the standard reference works.

174. Alignment of Supports. In §§ 168–173, it has been assumed that all of the supports are at the same level. In more extended texts, equations are derived to show, quantitatively, the effect produced by a difference in level of the supports on the amounts of the shears and moments. Without attempting such a derivation, we can see the general effect of changing levels for the supports by reference to Fig. 301, which shows a continuous beam of two spans, resting on the level supports R_1, R_2, and R_3. If R_2 is gradually lowered, as shown by dotted lines, the stiffness

of the beam will slowly transfer the load to the outside supports until at last all of the load will be carried on R_1 and R_3. In such a case, the beam becomes a simple beam of the span $L_1 + L_2$ and the bending moments and shears are very much increased. A similar thing occurs if R_2 is raised above R_1 and R_3. Hence, if a beam is to be designed as a continuous beam with the reduced bending moments that result from continuity, it is important that the designer be well assured that there is no possibility of settlement in any of the supports.

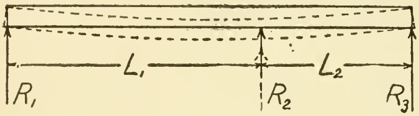


Fig. 301

175. Bending Moments Used in Design. The loading in any structure consists of the following elements: (*a*) The weight of the structure itself, called the ***dead load.*** This load is always in place and is present throughout the structure. (*b*) All loading other than that due to the weight of the structure itself, called the ***live load.*** This is usually present in varying amounts in different parts of the structure, and varies from time to time, and from place to place.

The usual procedure is to design a structure to carry the dead load plus an estimated live load uniformly distributed throughout the structure. When only simple beams are used, this procedure is satisfactory; but when continuous beams are used, the maximum moments and shears do not occur under a full live load. In general, the maximum moments occur when each alternate span is fully loaded and those between are without live load.

It is obvious that in an actual building, the placing of the live load cannot be accurately foreseen. It therefore becomes necessary to estimate what is apt to be the maximum moment produced by loading. In reinforced concrete practice, this subject has received much attention.

The following summary gives the amount of the maximum bending moments which are commonly assumed to be a proper basis of design.

(*a*) For continuous beams of several spans, and with the outside ends restrained, the maximum positive moment at the center and the maximum negative moment over the supports is taken at $WL/10$ for the end span and $WL/12$ for interior spans.

(*b*) For beams continuous over two spans only, the maximum positive moment is taken at $WL/10$ and the maximum negative moment is taken at $WL/8$.

PROBLEMS

Determine the reactions, shears, bending moments, and points of contraflexure in the following cases of continuous beams.

(1) Two equal spans with equal concentrated loads at the centers of the spans.

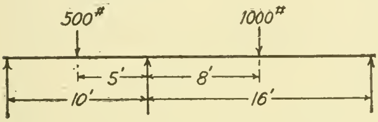


Fig. 302

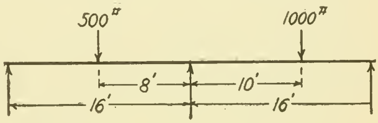


Fig. 303

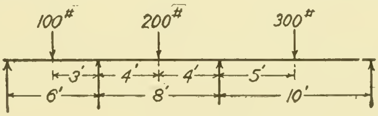


Fig. 304

(2) Three spans, as in Problem 1.

(3) Fig. 302.

(4) Fig. 303.

(5) Fig. 304.

CHAPTER XIX

COLUMNS

176. Introduction. The support shown in Fig. 24 will be recognized as a *column,* while that in Fig. 185 is called a *pier,* and that in Fig. 186 is called a *bearing block.* The function of each of them is to carry a load producing compression. The distinction between them lies in the proportion between length and cross section. Commonly, the term ***column*** or ***strut*** is applied to a more or less slender member which stands in a vertical position and carries a load from above. But in this discussion the term is applied to any relatively slender member stressed in compression, regardless of its position.

The compressive or crushing strength of a material, as ordinarily quoted, is determined from experiments on relatively short thick specimens. If these experiments are repeated on long thin pieces of the same material, a remarkable falling off in *unit* strength is noted. For example, take an ordinary yard stick measuring $1\frac{1}{8}'' \times \frac{3}{16}''$ and made of spruce. Its ultimate strength in compression, if computed from ordinary compressive strength data, would be

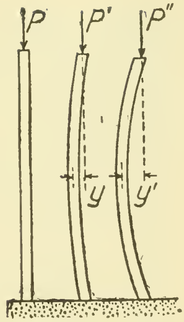
$$1\tfrac{1}{8}'' \times \tfrac{3}{16}'' \times 4{,}800 \text{ lbs./sq. in.} = 1{,}000 \text{ lbs. approx.}$$

But a load of about 10 lbs. would be found to be as much as the yard stick could actually carry in compression.

This large discrepancy can easily be accounted for if we note what happens to the long thin piece as the load is gradually applied. Under very light loads the piece remains straight (as in Fig. 305A) but soon it begins to bend. If the load which first produces bending* (as P', Fig. 305B) is left constant, the piece will support it indefinitely and the deflection y produced by bending also will remain fixed.

But if the load P' is *increased* to P'' (Fig. 305C), the bending stresses (due to the moment $P'y$, Fig. 305B) increase, and in turn

* Called the *critical load.*



A B C

FIG. 305

produce a greater lever arm y' (Fig. 305*C*). The total moment producing bending stress is now $P''y'$, which will go on increasing the lever arm, and thereby its moment, until failure soon follows.

Moreover, even though the ***critical load*** P' is not increased, experience and theory alike show that the slightest application of horizontal force, or even a slight and accidental jarring is enough to destroy the unstable condition of equilibrium, since this increases the lever arm of the load and finally results in failure. The critical load is then, to all intents and purposes, the ultimate load.

The distinction between a column and a compression piece lies in the tendency of the column to bend under axial loads. This tendency to bend varies in a general way with the proportion between the length of the column and the *least* dimension of its cross section. When this proportion is large (the case of a very slender column), the tendency to bend is large, and failure, when it occurs, is due almost wholly to bending. When this proportion is small (as in the case of a short block), the tendency to bend is negligible, and failure is due to crushing only. For intermediate cases (most columns in actual use come in this class) failure is due to a combination of crushing and bending in an indeterminate proportion.

177. Causes of Bending in Columns. There are four main causes for the bending which occurs or tends to occur in a loaded column. They are all concerned with the necessary inaccuracies of even the best of workmanship. Theoretically, a column might be made

(1) perfectly straight and of uniform cross section,
(2) with ends perfectly formed and aligned,
(3) of perfectly homogeneous material,
(4) loaded exactly along the axis.

Such a column would be an ***ideal column*** and would not tend to bend. But no real column can be made to these specifications.

Therefore, all real columns do tend to bend, since unavoidable imperfections tend to cause unequal stress distribution on the various cross sections. These stresses in turn result in unequal deformations. The natural result of unequal deformation is to cause a curvature which indicates the presence of bending stresses.

An ideal column would fail by crushing, at a unit stress determined by the ultimate compressive strength of the material. A real column fails by bending or by a combination of bending and crushing. The actual maximum unit stress which occurs in such a case is not readily determined ((1) p. 269) but it is clear that the *average* unit stress P/A will be less than the crushing strength of the materials by some amount which will depend on how great a tendency toward bending is present in a given case.

It should be particularly noted that the presence of bending stress is due to *accidental* variations from ideal conditions. Since these variations *are* accidental, it is impossible to tell to what extent and in what proportion any one of them influences a given case.

Furthermore, it is not possible to determine to what extent failure in a given case is due to bending or to crushing except in the cases of very long columns or short blocks. From these considerations, it follows that any attempt to construct a rational column formula for any except very long columns (in which case the failure is due to bending only) is useless, and that our knowledge of the strength of columns must be obtained very largely from results of experiments.

178. Classification of Columns. A. Type of Failure. The considerations outlined in the previous paragraph and a study of actual cases leads to a classification of compression members based on the manner of failure. We have seen that long slender columns fail by bending; that short stocky pieces fail by crushing; and that for intermediate cases failure is due to a combination of bending and compression. But since the very terms long and short, slender and stocky, are relative, and merge insensibly one into another, we must not expect that any such classification will or can be definite, nor that one type of failure can be infallibly

distinguished from another except in the extreme cases. There must be then a certain arbitrary element in any such classification. Moreover it should be understood that the words long and short, when applied to columns, do not refer to actual lengths but rather to length *relative to* cross-sectional dimension.

General practice seems to set up the following approximate classifications.

Short blocks or piers. These vary from a cube to a prism about 15 diameters high.

The shorter lengths are strictly short blocks and they fail by crushing (or more strictly speaking by shear, see § 78). The longer lengths, sometimes called *piers, posts,* or *struts,* show a somewhat smaller unit strength than the short blocks.

Columns. These vary in length from 15 to say 40 or 50 diameters. They fail from a combined crushing and bending. Their *unit* strength varies quite definitely relatively to their proportions and becomes less as the column becomes more slender.

Long columns. These are over say 45 diameters in length and fail in bending. Their *unit* strength is quite small in comparison with that of a short block and becomes less as the slenderness increases.

We have here a condition not before encountered as the *unit* strength varies with the proportions of the piece. Figure 306

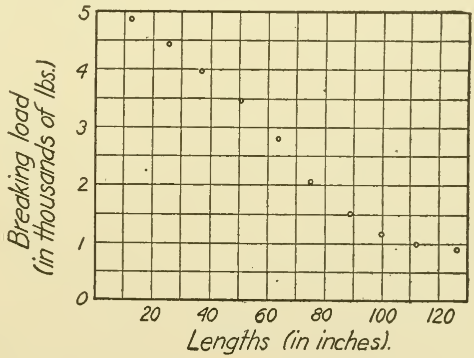


FIG. 306

Results of tests on $1\frac{3}{4}'' \times 1\frac{3}{4}''$ spruce struts. Each dot represents the average of five tests. (From U. S. Bureau of Standards; T. P. 152.)

has been condensed from a bulletin of the Bureau of Standards to show how the unit strength of a $1\frac{3}{4}'' \times 1\frac{3}{4}''$ spruce strut varies with its length.

B. Shape of End Bearings. We have seen how the amount of bending which occurs influences the unit strength of a column. But the tendency of a column to bend may be influenced, not only by its slenderness, but also by the form of the ends. Thus in Fig. 307 is shown a diagrammatic representation of four types of end conditions. In A the ends are definitely fixed so that the tangent to the elastic curve produced by the bending of the column is vertical. In B the ends of the column are spherical. In C is a combination of cases A and B. Evidently the rounded ends will allow bending to occur more freely than the fixed ends. Consequently the strength of the column in B will be less than that in A, with an intermediate value for the column in C. This phenomenon is exactly parallel to the strengthening and stiffening of a beam which occurs when its ends are restrained, which is explained in § 167.

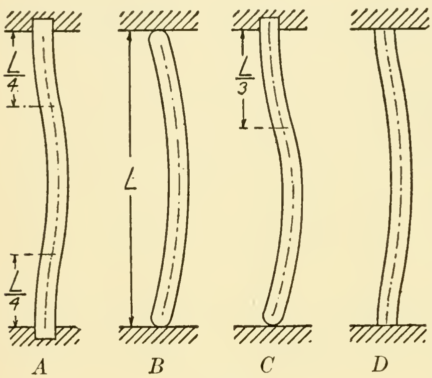


Fig. 307

Figure 307D shows a "flat" end condition. Here the ends are merely cut square and inserted between the bearings without any attempt at fastening. This gives about the same resistance to bending as the case shown in Fig. 307C.

In practice fully rounded or absolutely fixed ends are encountered rarely, if at all. Deformation of the supports usually will prevent absolute fixation (compare with § 161) and friction will prevent absolute freedom in turning. However, these cases are important to the theory of columns as they fix definite limits to the possibilities.

In Fig. 308 are shown some common types of column ends

encountered in practice. The flat end (Fig. 308*B*) is the most usual case. The ends of the column are cut square but no particular attempt is made to prevent movement. The effectiveness of a flat end in preventing rotation obviously depends on the

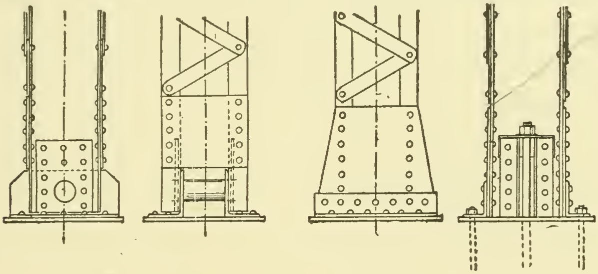


Fig. 308

size of the bearing in relation to the cross section of the column, and on the accuracy with which the plane of the column end is fitted to that of the bearing. In the case of long columns, flat ends show by test about the same strength as round ends, while in the case of a short column, flat ends show nearly as great strength as fixed ends (see curves *C* and *D*, Fig. 309). A fuller discussion of the effect of end conditions will be found in § 181.

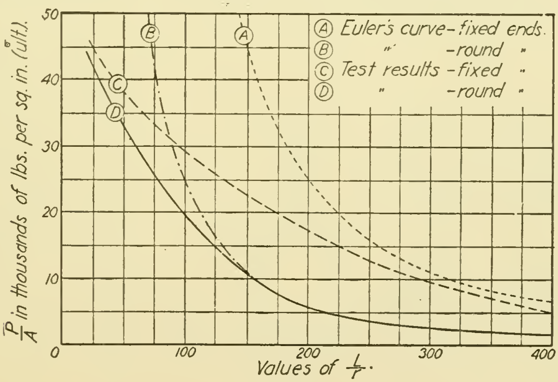


Fig. 309

Curves showing variation in the ultimate unit strength of columns having different end conditions and different proportions of length to cross section. Ordinates show unit strengths. Abscissas show degree of slenderness. Curves *A* and *B* were determined theoretically. Curves *C* and *D* were determined experimentally.

179. Characteristic Shapes. The resistance of a piece of material to crushing (so long as the load is concentric) is determined solely by its material and the cross-sectional area (§ 56). Resistance to bending, however, depends not only on the material and *size* of the cross section, but also quite largely on the *shape* of the section (§ 135). We may expect therefore that in a column (because of its tendency to bend) the shape as well as the size of the cross section will be important.

A beam usually tends to bend in a definite direction; therefore its ideal shape is so disposed as to have a large moment of inertia about an axis perpendicular to that direction. On the other hand, the bending of a column (since it originates in accidental causes) is apt to take place in any direction. Hence the ideal cross section for a column is a circle, since a circle has the same moment of inertia about any axis passing through its center. Again, since resistance to bending depends on moment of inertia, and since material near the axis does not contribute largely to moment of inertia, it will be evident that a hollow circle is about the ideal column cross section. Figure 310 shows characteristic column sections in three different materials. It will be seen at once that they are all more or less successful attempts to equalize the moment of inertia about all axes and to avoid material placed close to the center.

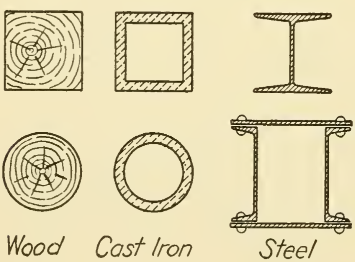


Fig. 310

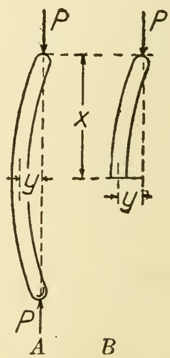


Fig. 311

As to the characteristic shape of a column along its length, consider Fig. 311, which shows a column with round ends bent

by the load P. If the top half is taken as a free body, as in Fig. 311B, it will be seen to be under a bending moment Py, as well as a direct stress P. But if we cut the top quarter free, the bending is seen to be less (since y is less at that section), and it would be found to decrease as the section is cut closer to the top (or bottom) end. But the end section (as well as all other sections) must carry the direct load P, and hence must be large enough to resist crushing due to P. The other sections must be large enough to resist this crushing as well as the bending which develops *at that section.* Hence the ideal shape for a column with rounded ends will resemble Fig. 312, the cross section being a hollow circle.*

FIG. 312

As a matter of fact, with the relatively short columns and low unit stresses in general use, actual deflections from a straight line do not occur. Hence it is not usual to use such shapes in ordinary practice, but they are sometimes seen in derrick booms, aeroplane struts, etc.

180. Column Formulas. In order to understand modern column formulas thoroughly, it is worth while to know something of the history of their development. The first column theory of importance was developed by Euler in 1757. At this time experimental investigations were sharply limited by a lack of suitable equipment. Hence Euler proceeded along rational lines after making certain assumptions. He assumed a column to fail by bending only,† and, working on that assumption, produced a formula which stands today as a proper solution for columns that do fail by bending, i.e., for very long columns (see § 176).

But when Euler's formula was put to an experimental test, it was found that, while it agreed with test results for long columns (which do fail in bending), for the shorter columns (which actually fail in part by crushing), it indicated too high an allowable load.

* In the case of a column with *fixed* ends the moment is as great at the ends as at the center. Therefore the end section should not have a reduced area.

† For a more complete statement of Euler's assumptions, see § 181.

This led to an attempt to correct Euler's formula by the insertion of certain arbitrary constants designed to make the amended formulas agree with experimental results (see § 183). As time went on and experimental methods were improved, it became more and more clear that in the ordinary cases, the crushing effect was so intermixed with the bending, and the causes of bending were so indeterminate that the wisest thing was to abandon the attempt to reach a rational solution and to frankly accept the experimental method. This has led to the modern formulas which are nothing more than the equations of curves determined by experimental means. (See § 184.)

Thus we have three classes of column formulas.*

(1) *Euler's formula.* This is a rational solution for the case of long columns only. It gives the ultimate strength of the column and is applicable to columns of any material (§ 181).

(2) *Modifications of Euler's formula.* These are intended to extend Euler's formula to the case of columns that fail partly by crushing. They use Euler's formula as a basis and modify it by inserting a constant term determined by experiment (§ 183).

(3) *Modern working formulas.* These are based almost entirely on results of tests and are stated in terms of the working unit stress instead of the ultimate stress. They are different for each material and are applicable only to columns having certain proportions of length to cross section (§ 184).

181. Euler's Theory for Long Columns. Euler's theory is based on the following assumptions.

(1) That the column is a "long" column, as defined in § 178; hence its failure is due to bending in so large a degree that the direct compressive stress produced by the load may be neglected *without sensible error.*

(2) That the unit stress in the column does not exceed the elastic limit of the material, (as determined by tests on short blocks), and that therefore the theory of deformations due to

* In more extended books on strength of materials can be found the secant formula, which is an attempt to develop a rational formula that can be generally applied. It is not, however, of sufficient importance, either in its theory or in practice, to justify its inclusion here as forming a fourth class.

bending may be employed. It will be shown later (see note, p. 278) that this assumption is a reasonable one.

(3) That the curvature of the column is slight, and that the cross section is constant throughout the length. This assumption is also necessary if the theory of deformations due to bending is to be used.

A. The General Theory for Columns with Round Ends. Let the column be loaded with its *critical load* (§ 176). If the column is a real column, this load will produce a slight bending as shown (much exaggerated) in Fig. 313A. If the column were an ideal one (§ 177), it would not bend, but the application of the least horizontal force or even a slight jar would cause bending to occur. In either event we may assume the column to be bent under the action of the load P, and to remain so while the load remains fixed. Any increase of the load will now cause failure, as shown in § 176. It is required to determine this critical load P, which can be taken as the ultimate load.

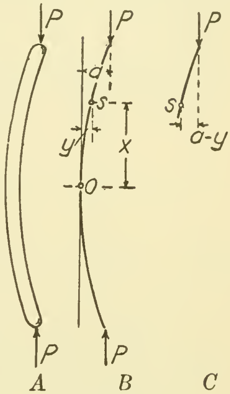


Fig. 313

Let Fig. 313B show the elastic curve of the column, referred to an origin of coordinates at o. The axis of y is taken horizontally so that our nomenclature will agree with that of Fig. 266 and § 152. The end of the column being round, our elastic curve is a simple curve, bending in the same direction throughout. For a definite column of length L, there will be a definite critical load P, and this load will produce a constant deflection a of the upper end of the column (where P is applied), away from the X axis (§ 176).

We may now write the equation of the elastic curve of the column (§ 152) in the form

$$M = P(a - y) = EI\frac{d^2y}{dx^2}\cdot \tag{1}$$

Multiplying each side of equation (1) by $2(dy/dx)$, we get

$$2P(a - y)\frac{dy}{dx} = 2EI\frac{dy}{dx}\frac{d}{dx}\left(\frac{dy}{dx}\right).$$

Integrating this expression, remembering that $dy/dx = 0$ for $y = 0$, we find

$$2ay - y^2 = \frac{EI}{P}\left(\frac{dy}{dx}\right)^2.$$

From this, we obtain

$$dx = \pm\sqrt{\frac{EI}{P}}\frac{dy}{\sqrt{a^2 - (a - y)^2}},$$

and, by a second integration, we find

$$x = \pm\sqrt{\frac{EI}{P}}\cos^{-1}\left(\frac{a - y}{a}\right), \tag{2}$$

the constant of integration being zero since $x = 0$ for $y = 0$. The upper sign is used for the upper half of the curve (Fig. 313), and the lower sign for the lower half.

Equation (2) is the equation of the elastic curve of the column. It holds good for *all* points on the curve. At the upper end of the column, x is approximately* equal to $L/2$, and y is equal to a. Substituting these values in (2), we have

$$\frac{L}{2} = \sqrt{\frac{EI}{P}}\cos^{-1}(0) = \sqrt{\frac{EI}{P}}\frac{\pi}{2},$$

$$P = \frac{EI\pi^2}{L^2}. \tag{3}$$

This gives the least value of P which will bend the column and keep it bent; in other words, it is the critical load.

In handling problems dealing with columns, it is more convenient to deal with unit stresses than with total loads. For this reason, equation (3) is usually changed in form by dividing through by the area A of cross section. This gives

$$\frac{P}{A} = \frac{EI\pi^2}{AL^2} = \frac{\pi^2 E}{\dfrac{L^2}{\dfrac{I}{A}}} = \frac{\pi^2 E}{\left(\dfrac{L}{\sqrt{\dfrac{I}{A}}}\right)^2}. \tag{4}$$

* By assumption (3), page 263.

In this expression, the quantity $\sqrt{I/A}$ is the factor which expresses the effect of the size and shape of the cross section (§ 179) just as the section modulus (§ 135) measures the value of a beam section. For convenience, it is given the special name ***radius of gyration*** (see § 182), and the special notation r. Using this notation, equation (4) reduces to the form

$$\frac{P}{A} = \frac{\pi^2 E}{\left(\frac{L}{r}\right)^2}. \tag{5}$$

This is Euler's formula for long columns with *round ends*. There are several points in connection with this expression that should be noted.

(1) The quantity P/A is not the *actual maximum* unit stress in the column; it is merely the total load divided by the cross-sectional area, i.e., a sort of *average* unit stress. The actual maximum unit stress is due to bending and cannot be derived from equation (3).

(2) The quantity (a) which was used to express the total deflection has vanished. Therefore P/A, as given in (5), may be regarded as that unit stress which will hold the column at *any small* deflection at which it may be set. It is therefore the average unit stress due to the critical load.

(3) The material of which the column is made enters this expression by way of E. Thus it is the *stiffness* and not the *strength* of the material that is important. This of course is because the whole theory had to do with the initial deformation which leads to failure.

(4) The dimensions of the column are represented by L/r. This expression is a relation between length and cross section, called the ***slenderness ratio.*** Both L and r are usually expressed in inches. They must always be expressed in the same units.

Every column *tends* to bend in a direction perpendicular to that axis about which the I is least. Therefore the r (which is derived from I), which enters the formula and which measures,

in part, the resistance of the column to bending, should be the *least* r for the section (see also § 226), since this r gives the least P.

B. Fixed Ends. The form of the elastic curve of a long column with round ends is shown in Fig. 314A. In Fig. 314B is a column with fixed ends. The form of this curve resembles in a general way that for a beam with fixed ends. At the points of inflection b and c, there is no moment and no shear. The thrust P is the only force acting, as shown in Fig. 314C. Therefore, the part bc is in the same condition of stress as a round-ended column. Also the part ab is in the same condition of stress as one half of bc. Hence the length $ab = \frac{1}{2}bc = \frac{1}{4}ad$. This means that the ultimate strength of a column with fixed ends is the same as that of a round-ended column of one half the length. Therefore, if we substitute $\frac{1}{2}L$ for L in equation (5), we will have the ultimate unit strength of a column with fixed ends, thus

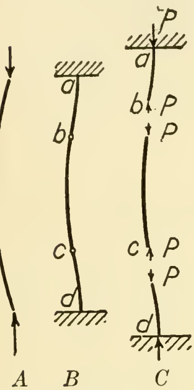


Fig. 314

$$\frac{P}{A} = \frac{4\pi^2 E}{\left(\frac{L}{r}\right)^2}. \tag{6}$$

C. Fixed and Round Ends. From considerations similar to those in B, we can arrive at the following expression to cover this case:

$$\frac{P}{A} = \frac{(9/4)\pi^2 E}{\left(\frac{L}{r}\right)^2}. \tag{7}$$

D. Flat Ends. There is no theoretical consideration that has led to a definite evaluation for the strength of a column with ends cut flat but not fixed (the ordinary condition in structural work). As shown by tests, the strength for flat ends approaches that for round ends, in the case of long columns. In the case of short columns, the strength for flat ends approaches that for fixed ends (Fig. 309 and § 178B).

E. General Form of Euler's Formula. Euler's formula is often written in the form

$$\frac{P}{A} = \frac{n\pi^2 E}{\left(\frac{L}{r}\right)^2}. \tag{8}$$

In this form n stands for a number which depends on the condition of the ends. In the following table the value of n for flat ends is one commonly used though it has no theoretical basis, as explained in D above. The other values are taken from equations 5, 6, and 7.

Values of n in Euler's Formula

Round	Fixed and Round	Flat	Fixed
1	$2\frac{1}{4}$	$2\frac{1}{2}$	4

F. Limitations. Column formulas are commonly expressed by means of a curve having values of P/A for the ordinates and L/r for the abscissas. Using Euler's formula as applied to steel columns (E = 25,000,000 pounds per sq. in.), the resulting curve is as shown in Fig. 309. Obviously the value of P/A approaches infinity as L/r approaches zero. But it is equally obvious that no material can sustain such unit stresses. The fallacy lies in the fact that Euler's formula is based on the assumption that failure is due to bending only. (See assumption (1).) In the case of the shorter columns, the direct thrust produces a crushing effect that is *not* negligible in comparison to the bending stresses. Therefore, Euler's formula gives results that are absurdly high in such cases. This is well illustrated by the curve for experimental results as given in Fig. 309. The point of departure between Euler's curve and the curve of experimental results (the ***limit of validity***) will vary with the material and the end conditions, but in general it will occur at a unit stress not greater than one third of the ultimate strength of the material in crushing (see § 183).

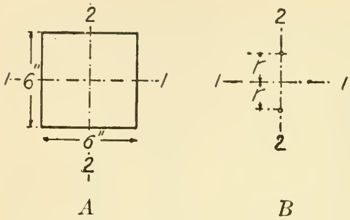


FIG. 315

182. Radius of Gyration. As noted on page 265, radius of gyration is a quantity which measures the effectiveness of a column section in the same manner that the section modulus I/c measures the effectiveness of a beam section.

Mathematically, radius of gyration can be expressed in the form

$$r = \sqrt{\frac{I}{A}}.$$

It is a quantity quite similar to moment of inertia, in that it comes into existence merely as a convenient way of expressing a frequently recurring relationship between other quantities.

Since the value of r is derived from that of I, it is obvious that r must be a quantity which is always related to an axis.

While radius of gyration is ordinarily large when the moment of inertia is large, this does not always hold good. By reference to a handbook, we find the I for a 12″ × 31.5 lb. I beam to be 215.8″^{4} and r is 4.83″. For the 12″ × 55 lb. I beam, $I = 321''^4$ and $r = 4.45''$. Thus as I increases, r decreases. This happens because of the fact that in this case the area increases faster than the value of I.

If we take any area, such as the square in Fig. 315*A*, and imagine all of that area concentrated at two points equi-distant from the axis *1–1*, as in B, and further impose the condition that I must be the same in A and B, then r is the radius of gyration of the square. For we have

in a: $$I = \int_A z^2 dA,$$

in b: $$I' = \int_A r^2 dA = r^2\left[\frac{A}{2} + \frac{A}{2}\right] = Ar^2,$$

since r is constant. But we have specified that $I = I'$. Then

$$Ar^2 = I,$$

whence

$$r = \sqrt{\frac{I}{A}}.$$

Obviously the same device could be employed in connection with any area and thus we see that the radius of gyration of any area is that distance from any given axis at which an entire area might be concentrated and have the moment of inertia remain the same as for the area itself.

The radius of gyration used in column formulas is usually the least radius of gyration for the given section (see page 266 and § 187).

PROBLEMS

1. What is the radius of gyration of a circle, about an axis through its center?
2. What is the radius of gyration of a hollow square, about an axis parallel to one of the sides and passing through the center? State the result in terms of the outside and inside diameters d and d_1.
3. In Problem 2, let the axis pass diagonally through the center.
4. Determine the radius of gyration of the I section, Fig. 347, about each of the axes of symmetry.
5. Find the radius of gyration of the section, Fig. 197*A*, about each of the given axes (*1–1* passes through the center of gravity).
6. What is the radius of gyration of the section shown in Fig. 130 about each of two rectangular axes passing through the center of gravity, one axis being parallel to the back of the channel?
7. Plot the curve of Euler's formula for timber columns with round ends and another for fixed ends. Let $E = 1{,}000{,}000$ lbs. per sq. in. Use 8″ × 10½″ cross section paper. Let the values of P/A vary between 0 and 6,000 lbs. per sq. in. and let those of L/r vary between 0 and 400.
8. What is the safe load on a timber column 6″ × 6″ in cross section and 14′ 0″ long? Round ends.
9. What, according to Euler's formula, is the safe load on a timber column 6″ × 6″ and 6′ 0″ long? Fixed ends.
10. Investigate the safety of a 6″ × 23 lb. H beam, 36′ 0″ long, when acting as a column with round ends and carrying a load of 25,000 lbs.
11. What is the safe load on a steel pipe whose diameters are 4″ outside and 3½″ inside, when acting as a column with fixed ends and which is 20′ 0″ long? Factor of safety 8.
12. What are the relative strengths of a wooden column 4″ × 4″ and another 4″ × 6″ if each is 20′ 0″ long and has flat ends?
13. What are the relative strengths of an 8″ × 34 lb. H beam and a 12″ × 35 lb. I beam when used as long columns?

183. Modifications of Euler's Formula. As pointed out above, Euler's formula does not give results that agree with tests, except

in the case of long columns (Fig. 309). This fact has led to many attempts to produce a rational formula that would agree with test results through a wide range of slenderness ratios. Three such formulas are given below.

A. THE GORDON OR RANKINE FORMULA. This is a partly rational formula as follows:

$$\frac{P}{A} = \frac{S}{1 + \phi\left(\frac{L}{r}\right)^2},$$

in which P/A is the average ultimate unit strength of a column, S is the ultimate crushing strength of the material, and ϕ is a constant designed to make the formula fit the results of experiments and which varies for each different material and each different end condition.

This formula is intended to apply to the entire range of values of L/r. When the values of ϕ are well chosen it does, roughly, carry out this intent but the values quoted for ϕ by various writers are quite unsatisfactory in many respects and the formula is not much used. The values for ϕ given in Merriman, *Mechanics of Materials*, are as follows:

Material	Both ends fixed	Fixed and round ends	Both ends round
Timber................	$\frac{1}{3{,}000}$	$\frac{1.78}{3{,}000}$	$\frac{4}{3{,}000}$
Cast Iron..............	$\frac{1}{5{,}000}$	$\frac{1.78}{3{,}000}$	$\frac{4}{5{,}000}$
Wrought Iron..........	$\frac{1}{36{,}000}$	$\frac{1.78}{36{,}000}$	$\frac{4}{36{,}000}$
Steel..................	$\frac{1}{25{,}000}$	$\frac{1.78}{25{,}000}$	$\frac{4}{25{,}000}$

A curve illustrating this formula *when adapted* to a column with flat ends and *with* a factor of safety applied is given in Fig. 316.

B. THE PARABOLIC FORMULA. From a study of the results of tests J. B. Johnson concluded that the strength of a column was

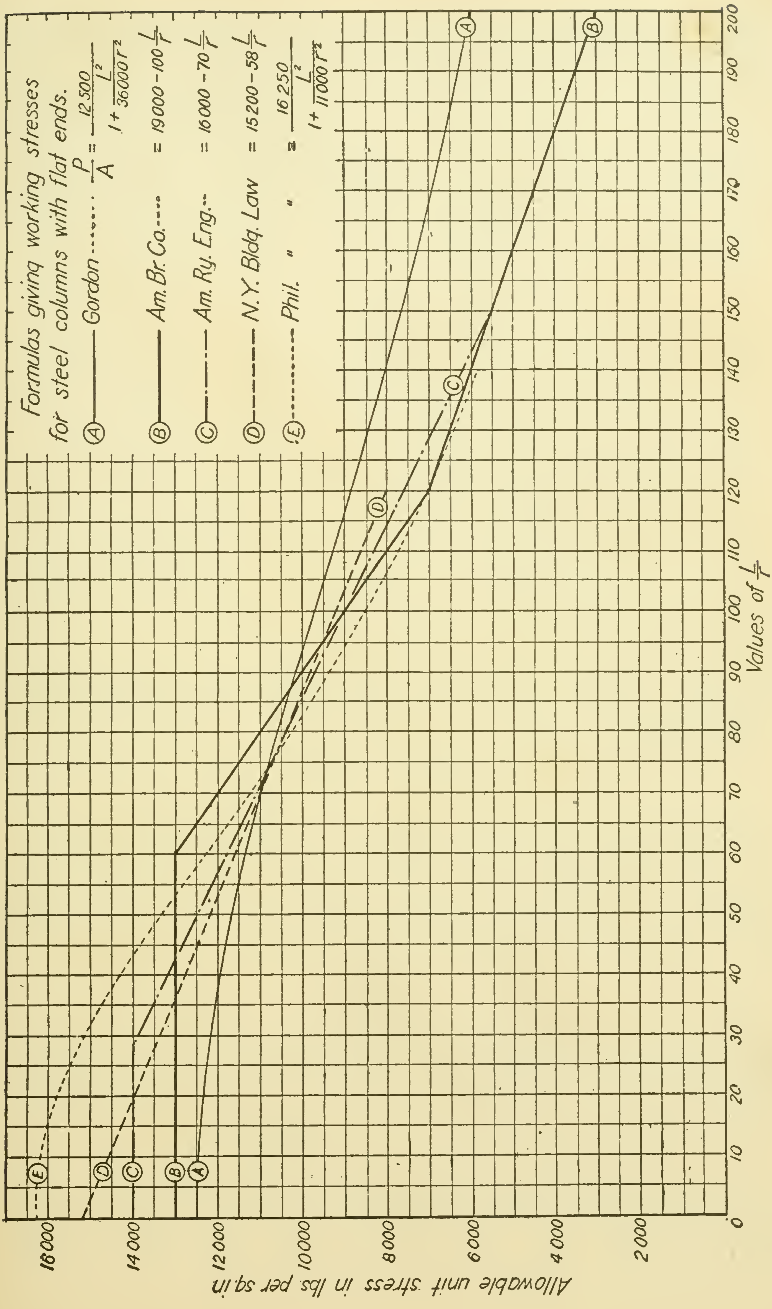

Formulas giving working stresses for steel columns with flat ends.
(A) Gordon $\frac{P}{A} = \frac{12\,500}{1+\frac{L^2}{36\,000\,r^2}}$
(B) Am. Br. Co. $= 19\,000 - 100\frac{L}{r}$
(C) Am. Ry. Eng. $= 16\,000 - 70\frac{L}{r}$
(D) N.Y. Bldg. Law $= 15\,200 - 58\frac{L}{r}$
(E) Phil. " " $= \frac{16\,250}{1+\frac{L^2}{11\,000\,r^2}}$
Allowable unit stress in lbs. per sq. in
Values of $\frac{L}{r}$
0 10 20 30 40 50 60 70 80 90 100 110 120 130 140 150 160 170 180 190 200
2 000 4 000 6 000 8 000 10000 12 000 14 000 16000

Fig. 316

limited by the yield point of the materials. This conclusion has since been supported by numerous independent tests on steel

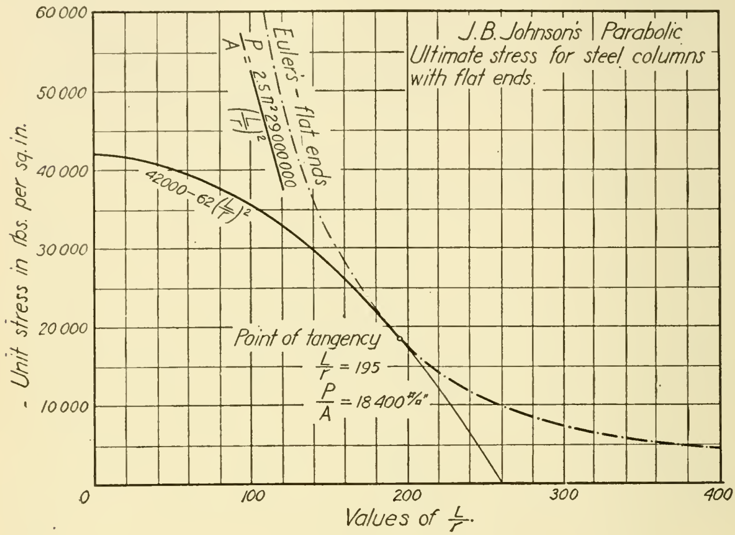


Fig. 317

columns. (§ 78.) The formula which he proposed as the result of his study is

$$\frac{P}{A} = S_y - \phi\left(\frac{L}{r}\right)^2$$

in which P/A is the average ultimate unit stress on the column, S_y is the yield point of the material, and ϕ is a constant which makes the curve come tangent to Euler's curve, as shown in Fig. 317. The value of ϕ is given by the equation

$$\phi = \frac{S_y^2}{4n\pi^2 E} \cdot$$

The coordinates of the point of tangency are given by the equations

$$\frac{P}{A} = \frac{S_y}{2}, \qquad \text{and} \qquad \frac{L}{r} = \left(\frac{2n\pi^2 E}{S_y}\right)^{1/2}.$$

The value of n in these expressions is the same as in Euler's formula, p. 267.

The point of tangency between this curve and Euler's curve marks the *limit of validity* for each formula. Since this formula gives ultimate stress, a factor of safety must be applied to permit its use in design.

C. THE STRAIGHT LINE FORMULA. The following formula was worked out by T. H. Johnson from test results. It has the advantage of simplicity, and it is perhaps as accurate as any.

$$\frac{P}{A} = S_u - \phi \frac{L}{r}$$

in which P/A is the average ultimate unit stress on the column, S_u is the ultimate compressive strength of the material, and ϕ

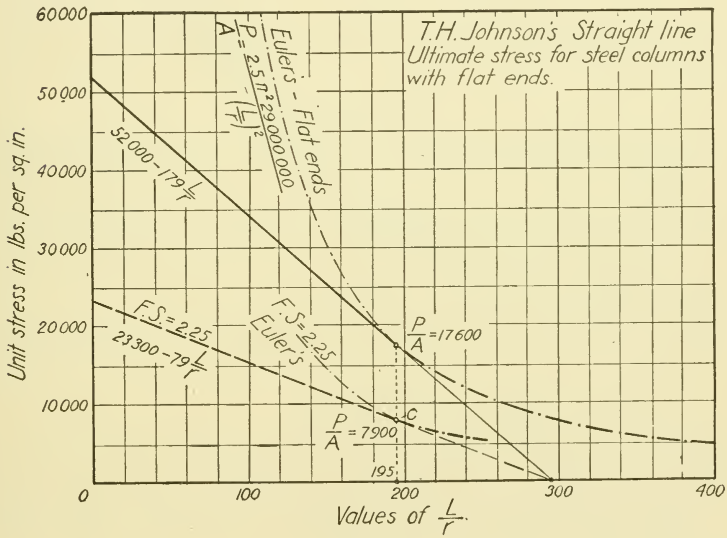


FIG. 318

is a constant which makes the line come tangent to Euler's curve, as shown in Fig. 318. The value of ϕ is given by the equation

$$\phi = (2/3)S_u \left(\frac{S_u}{3n\pi^2 E} \right)^{1/2}.$$

The coordinates of the point of tangency are given by the equations

$$\frac{P}{A} = \frac{S_u}{3}, \quad \text{and} \quad \frac{L}{r} = \left(\frac{3n\pi^2 E}{S_u}\right)^{1/2}.$$

The value of n in these expressions is the same as in Euler's formula, page 267.

The point of tangency between this line and Euler's curve marks the *limit of validity* for each formula. This formula gives the ultimate unit stress on a column of any material and for any specified end condition. If it is desired to apply any given factor of safety, this can be done graphically, as shown in Fig. 318 by the dash line. The equation of this line then becomes a working strength formula with its limit of validity at c.

NOTE. The limit of validity for Euler's formula as given in either B or C above is less than the elastic limit. Therefore assumption (2), § 181, is seen to be justified.

PROBLEMS

√ 1. What is the ultimate load on a 10″ × 10″ timber column with flat ends and 18′ 0″ long, as determined by each of the formulas of §§ 183B and 183C?

2. What load can be carried safely on a 5″ × 18.7 lb. H beam acting as a column 10′ 0″ long with flat ends? Factor of safety to be 3. Use the parabolic formula.

3. Solve Problem 2 using the straight line formula.

4. Derive a straight line formula for a steel whose ultimate strength is 45,000 lbs. per sq. in. and whose E is 29,000,000 lbs. per sq. in. Let the formula be arranged to show the working unit stress on a column with flat ends, allowing a factor of safety of 3.

5. What is the limit of validity for the formula derived in Problem 4?

6. Derive a parabolic formula for a timber whose yield point in compression is 2,800 lbs. per sq. in. and whose E is 1,300,000 lbs. per sq. in. Let it show the working unit stress on a column with flat ends when a factor of safety of 3 is allowed.

7. What is the limit of validity of the formula derived in Problem 6?

8. Derive a straight line formula for flat end columns made of cast iron having an ultimate strength of 50,000 lbs. per sq. in. and an E of 15,000,000 lbs. per sq. in. Allow for a factor of safety of 8.

9. Show a complete derivation for the values of ϕ and of the coordinates of the point of tangency for Johnson's straight line formula, as given in § 183C.

10. Show a complete derivation for the values of ϕ and of the coordinates of the point of tangency for Johnson's parabolic formula as given in § 183B.

184. Results of Tests. In Fig. 309 we have had occasion to show some curves which give the results of tests on steel columns. It should not be assumed that any set of tests covering a considerable number of cases will give results that can be represented *fully* by a smooth curve. When a number of pieces of any given slenderness ratio are tested, each piece will show a somewhat different strength from the others.

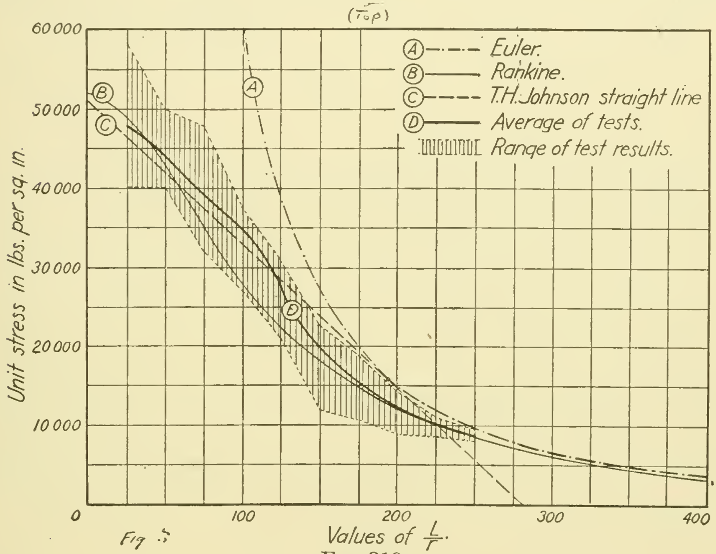


FIG. 319

Figure 319 has been adapted from a Bureau of Standards report on tests of spruce struts. It shows the variation of results obtained in a carefully conducted set of tests. Several typical column formulas have been plotted along with the test results to show how nearly the formulas conform to the results. From the figure it is obvious that any attempt at precision is useless, that a well-adjusted formula will give safe results, and that there is little justification for attempting to figure the strength of a column too closely or to make too fine a discrimination between the various formulas.

185. Working Formulas. The formulas so far discussed give the unit stress which causes *failure* in a column. But it is ordi-

narily more convenient to use an equation stated in terms of an *allowable* unit stress. Therefore working formulas are stated in that manner.

The three formulas discussed in § 183 as well as all those that follow can be conceived as being in one fundamental form, viz.,

$$\left\{\begin{array}{l}\text{The ultimate (or}\\ \text{safe) unit stress}\\ \text{on a column}\end{array}\right\} = \left\{\begin{array}{l}\text{The approximate}\\ \text{ultimate (or safe)}\\ \text{unit crushing}\\ \text{strength of the}\\ \text{material}\end{array}\right\} - \phi \left\{\begin{array}{c}\dfrac{L}{r}\\ \text{or}\\ \left(\dfrac{L}{r}\right)^2\end{array}\right\},$$

in which ϕ is a factor which, in the investigator's judgment, is best adapted to make the equation fit the known facts.

Working formulas are based almost entirely on tests and their limits of validity usually are clearly stated. The shorter columns (L/r between 0 and about 50) seem to have a fairly uniform strength, as indicated by the horizontal lines at the left of two of the curves in Fig. 316. Most engineers hesitate to design columns with L/r greater than about 150. This is not because more slender columns cannot carry a proper load but because, when columns are very slender, a small horizontal force produces large bending stresses, so that such columns are very susceptible to accidental damage. It is therefore usual to limit the use of a given formula, as shown by the stopping of the curves C and D in Fig. 316. Working formulas for steel, timber, and cast iron are discussed below.

A. For Steel Columns. The following formulas for the allowable unit stress on steel columns with flat ends are in general use:

(1) * *The American Bridge Company's formula:*

$$\frac{P}{A} = 19{,}000 - 100\frac{L}{r}.$$

This formula is limited in application to values of L/r between 60 and 120. For values less than 60 use 13,000 lbs. per sq. in. as the allowable unit stress (see Fig. 316). The specifications of which

this formula is a part do not allow the design of main compression members having slenderness ratios higher than 120. But *secondary* members may be designed for ratios of L/r between 120 and 200, by use of the formula:

$$\frac{P}{A} = 13{,}000 - 50\frac{L}{r}\cdot$$

(2) *American Railway Engineering and Maintenance of Way Association formula:*

$$\frac{P}{A} = 16{,}000 - 70\frac{L}{r}\cdot$$

This formula is limited to values of L/r between 30 and 150. For values less than 30 use 14,000 lbs. per sq. in. as the allowable unit stress (see Fig. 316).

(3) *The New York Building Law formula:*

$$\frac{P}{A} = 15{,}200 - 58\frac{L}{r}\cdot$$

This formula is limited to values of L/r less than 120 (see Fig. 316).

(4) *Gordon's formula with factor of safety:*

$$\frac{P}{A} = \frac{12{,}500}{1 + \dfrac{L^2}{36{,}000r^2}}\cdot$$

(5) *Philadelphia Building Law formula:*

$$\frac{P}{A} = \frac{16{,}250}{1 + \dfrac{L^2}{11{,}000r^2}}\cdot$$

This formula is limited to values of L/r less than 140.

From Fig. 316 it will be seen that all these formulas fall quite close together through the usual range of values of L/r, that is, between about 50 and 150. At present the straight line formulas are most used, as being simpler and quite as reliable as the more

complex forms. In all these formulas, of course, r stands for the *least* radius of gyration, as explained on page 269.

B. For Timber Columns. Wooden columns are usually round or rectangular in section. Therefore it is possible to express r in terms of the diameter or *least* side. It is thus possible to state the formulas in simpler terms than those used for steel. The formula proposed by the American Railway Engineering and Maintenance of Way Association is

$$^{*}\frac{P}{A} = S\left(1 - \frac{L}{60D}\right),$$

where S is the allowable crushing unit stress on the material and D is the least side of a rectangular section or the diameter of a circular section. This formula is limited to values of L/D between 15 and 30.

C. Cast Iron Columns. These are usually made with hollow circular or rectangular sections. The formulas are usually stated in terms of L/r though sometimes the L/D form is used.

The Chicago building law gives the formula

$$^{*}\frac{P}{A} = 10{,}000 - 60\frac{L}{r}\cdot$$

This is limited to values of L/r less than 70.

186. Use of Working Formulas. In using the various formulas found in books of reference, a number of points should be kept in mind.

1. *End conditions.* Most of the formulas are written for columns with flat ends though this is not always the case, particularly with the Euler, Gordon, and Rankine formulas.

2. *Limits.* Most of the modern formulas are frankly empirical, based on a definite set of tests. Therefore, they are applicable only to columns having a slenderness ratio that falls within the scope of the tests. Every column formula should have its limits of validity stated as a part of the formula, but unfortunately this is often neglected.

3. *Radius of gyration.* Every column tends to bend in a direction perpendicular to that axis about which the radius of gyration is least. Therefore, the r which appears in the formula should be the least possible r for the section concerned. (For an exception to this, see § 188.)

4. *Length.* The length of a column is counted as the greatest length between points of lateral support. It is stated in the same terms as r, usually in inches. (See also § 188.)

5. *Loading.* In all of the formulas, it is assumed that the loading is concentric: i.e., that there is no definite or determinable eccentricity, but only such as is accidental (§ 177, page 254). For the treatment of definitely eccentric loads, see § 194.

6. *Choice of formula.* There are so many working formulas in current use that the beginner is at a loss to make a choice. In many cases, the work to be done is governed by building laws or design specifications. For the problems in this book, the formulas given in § 185 and marked with an asterisk may be used.

The following illustrations are given for steel columns with flat ends.

A. Investigation. (1) Given an 8″ × 34 lb. H beam, 14′ 0″ long, acting as a column and carrying a load of 90,000 lbs. Is it safe?

By reference to a handbook, the section is found to have a *least* radius of gyration of 1.87″ and an area of 10 sq. in. Therefore, the slenderness ratio is (14 × 12) ÷ 1.87 = 90. This is within the limits of the American Bridge Company formula, which can be used. The allowable unit stress on this column as given by that formula is 19,000 − (100 × 90) = 10,000 lbs. per sq. in. The allowable total load is 10,000 lbs. per sq. in. × 10 sq. in. = 100,000 lbs. This shows the column is safe.

(2) Given a 6″ × 23.8 lb. H beam acting as a column 20′ 0″ long and loaded with 35,000 lbs. Is it safe? The least $r = 1.45''$ and the area = 7.00 sq. in. The slenderness ratio is (20 × 12) ÷ 1.45 = 166. This is beyond the limits of most column formulas. But if there is little danger of accidental damage (§ 185), the American Bridge Company formula for secondary members may

be used. The allowable unit stress is 13,000 − 50(166) = 4,700 lbs. per sq. in. The allowable total load is 4,700 × 7 = 32,900 lbs., which is less than the given load. Consequently, the column is *unsafe.*

(3) Given a 5″ × 18.7 lb. H beam acting as a column 25′ 0″ long and loaded with 10,000 lbs. Is it safe? The least $r = 1.20''$; $A = 5.50$ sq. in.; $L/r = 250$. This is beyond the limit of any accepted working formula, but under exceptional conditions (see 2 above) Euler's formula may be used. For a steel column with flat ends and $L/r = 250$, this formula becomes

$$\frac{P}{A} = \frac{2\frac{1}{2}\pi^2(29{,}000{,}000)}{(250)^2} = 11{,}400 \qquad \text{lbs. per sq. in.,}$$

which is the ultimate unit stress. The ultimate load is 11,400 × 55 = 62,7C0 lbs. The factor of safety should depend on the conditions of the special case, say 8.* Hence the allowable load is about 8,000 lbs. The column is *unsafe* for 10,000 lbs. load and a factor of safety of 8.

B. Safe Load. The application of column formulas to the determination of the safe load does not involve any principle not noticed above.

C. Design. In the case of design the loading, length, and material usually are given and the cross section is to be determined. Therefore, the ordinary column formula will contain two unknowns (A and r) and cannot lead to a solution unless these unknowns can be reduced to common terms. In the case of solid square or circular sections (such as are usual for wooden columns) this can be done. But in the case of steel shapes it cannot be done. Therefore, we are forced to use the method of trial and error. This means that a section is assumed and the case investigated as in (A) above. If it does not prove satisfactory, another section is chosen, *using the results of the first computation* as a guide. This process is repeated until a satisfactory section is found.

* Because very long columns are peculiarly susceptible to accidental damage (§ 185), factors of safety for such cases should be especially generous.

Let it be required to design a steel column 7′ 0″ long and to carry 60,000 lbs. Moreover, let it be required that the column be made from an I beam and that the column be designed by use of the American Bridge Company formula.

From the limits of validity of the formula, we know that our unit stress must be between 13,000 lbs. per sq. in. and 7,000 lbs. per sq. in. Therefore, the area of our section must be between 60,000/13,000 = 4.6 sq. in. and 60,000/7,000 = 8.5 sq. in. roughly. From a table of elements of beam sections, it is evident that our column will not be smaller than a 7″ I nor larger than a 12″ I. Let us choose an 8″ × 18 lb. I as a trial size.

The least $r = 0.84''$; $A = 5.33''$; $L/r = 100$, and $P/A = 19{,}000 - (100)(100) = 9{,}000$ lbs. per sq. in. Therefore, the allowable load $= 9{,}000 \times 5.33 = 48{,}000$ lbs., which is insufficient. If we make one more computation, using a slightly larger section, the result should be found.

187. Built-up Sections. Steel sections are not manufactured in sizes large enough to serve for heavily loaded columns. Therefore, it is common practice to connect two or more pieces by means of plates or lattice bars, as shown in Figs. 310 and 320. The plates or bars are riveted to the main members and cause the separate sections to act as one; i.e., the separate parts are prevented from bending independently in a direction perpendicular to the axis about which their radii of gyration are least.

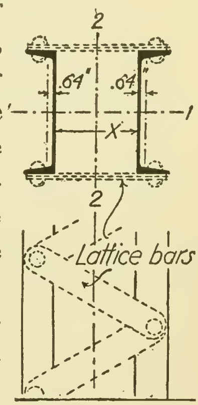


FIG. 320 Built up column composed of two 10″x 15 lb. channels latticed together.

Thus in Fig. 320 is a column built up of two 10″ × 15 lb. channels, latticed together. The strength of the column is determined by the radius of gyration of the two sections (considered as one) about axis *1–1* or *2–2*, whichever is least. It is usual to make the distance x such that the two moments of inertia (and therefore the two radii of gyration) are equal. Lattice bars are not counted as an effective part of the section. In the

given case, the distance x is determined as follows:

$$I \text{ about axis } 1\text{–}1 = 2 \times 66.9 = 133.8,$$
$$I \text{ about axis } 2\text{–}2 = 2(2.3 + 4.46((x/2) + 0.64)^2).$$

Equating these two values and solving, x is found to be about 6 in.

188. Braced Columns. The length of a column is taken as the distance between supports which furnish effective bracing against sidewise deflection. In multiple-story skeleton-framed buildings, L is taken as the distance between floors; as the floors furnish the requisite bracing.

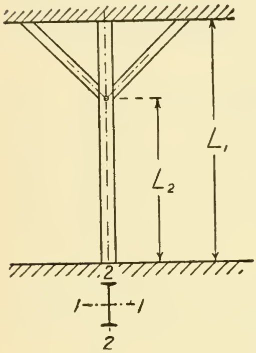


Fig. 321

Sometimes a column is braced in one direction only. Thus in Fig. 321, the column is braced by two struts in the plane of the paper, and hence is free to deflect throughout its length in only one direction, viz., perpendicular to the axis *1–1*. In the other direction it is free to deflect through the length L_2 in either direction, but it would naturally deflect perpendicular to axis *2–2*. In such a case, the slenderness ratio is either L_1/r_1 for axis *1–1* or L_2/r_2 for axis *2–2*, whichever is greatest.

189. Relation of Working Formulas to Euler's Formula. In §§ 185 and 186 it was shown that it is the usual practice to limit the application of working formulas to columns having ratios of L/r below or between certain values. If the conditions warrant it, in a given case (§ 186), it may become desirable to design a column whose L/r is greater than the limit set by a given formula. In that event it is desirable to extend the working formula by a curve derived from Euler's formula by applying a factor of safety. In order that this may be done with no break in the continuity of the formulas, it is necessary to have the curve derived from Euler's curve come tangent to that of the working formula. But since Euler's curve is for ultimate strengths, and

since the factor of safety used in a given working formula is rarely stated, it becomes necessary to determine the factor of safety in each case.

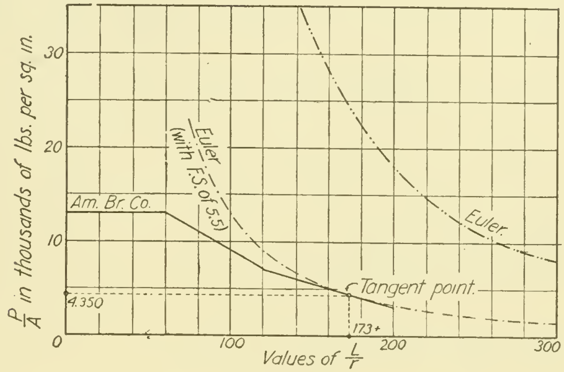


FIG. 322

As an illustration, let the American Bridge Company formula for secondary members be chosen (§ 185). This formula is for steel columns with flat ends and its curve is shown in Fig. 322. The curve of Euler's formula for steel columns with flat ends is also shown. Now evidently by applying some (unknown) factor of safety to the Euler curve, a derived curve can be found which will come tangent to the sloping line of the A. B. C. formula. Let it be required to determine the factor of safety.

For convenience let values of P/A be written as y and values of L/r be written as x.

Then Euler's formula becomes $y = n\pi^2E/x^2$, and when a factor of safety (f) is applied the equation of the derived curve will be

(1) $$y = \frac{n\pi^2E}{fx^2}.$$

The equation of the A. B. C. formula is then

(2) $$y = 13{,}000 - 50x.$$

In equation (1)

$$\frac{dy}{dx} = -\frac{2n\pi^2Ex^{-3}}{f}.$$

In equation (2)

$$\frac{dy}{dx} = -50.$$

If the two curves are to be tangent, these slopes are equal; that is,

$$\frac{-2n\pi^2 E}{f} x^{-3} = -50.$$

Hence

$$\frac{n\pi^2 E}{f} = 25x^3. \tag{3}$$

Substituting this value in (1), we find

$$y = 25x.$$

But (1) and (2) give the same y. Hence

$$\begin{aligned} 13{,}000 - 50x &= 25x, \\ x &= 173 \text{ (about)}. \end{aligned}$$

Then, from (3), using this value of x, we have

$$\begin{aligned} f = \frac{n\pi^2 E}{25x^3} &= \frac{(2\tfrac{1}{2})\pi^2(29{,}000{,}000)}{25(173)^3} \\ &= 5.5 \text{ (about)}. \end{aligned}$$

These relations are illustrated in Fig. 322.

NOTE. In all of the formulas developed and quoted in this chapter, we have assumed that the loads are concentric or nearly so. That is to say, that there is no definitely determinable eccentricity. For definitely eccentric loading, the treatment outlined in § 194 should be used.

PROBLEMS

NOTE. In the following problems let all columns have flat ends.

1. What is the safe load on a timber column 6″ × 6″ and 10′ 0″ long?
2. (*a*) What is the safe load on a 10″ × 25 lb. I beam acting as a column 20′ 0″ long? (*b*) If the above column were 9′ 0″ long, what would be the safe load? (*c*) If the above column were 15′ 0″ long, what would be the safe load?
3. Investigate the safety of a cast iron column, 10″ in diameter outside and 8″ inside, and 14′ 0″ long, when carrying a load of 100,000 lbs.
4. Investigate the safety of a timber column, 8″ × 8″ and 14′ 0″ long, when carrying a load of 50,000 lbs.

5. Investigate the safety of a timber column, 6″ × 8″ and 11′ 0″ long, carrying a load of 23,000 lbs.
6. What is the safe load on a cast iron column 8″ × 10″ outside and $6\frac{1}{2}$″ × $8\frac{1}{2}$″ inside; length 12′ 0″?
7. What is the safe load on a timber column 6″ × 6″ and 30′ 0″ long? Let the factor of safety be 5.
8. If the column in Problem 7 is 18′ 0″ long, what is the safe load?
9. Design a steel column 14′ 0″ long to carry a load of 73,000 lbs.
10. Design a timber column 12′ 0″ long to carry a load of 50,000 lbs.
11. Design a steel column 20′ 0″ long to carry a load of 30,000 lbs.
12. Design a wooden column 20′ 0″ long to carry a load of 8,000 lbs.
13. A column is to be built up of two 15″ × 37.5 lb. I beams latticed together as in Fig. 320; at what distance on centers should they be placed?
14. A column is to be built up of two 15″ × 40 lb. channels and solid plates $\frac{5}{8}$″ thick in the place of latticing. What should be the width of the plates (approximately)?
15. If the column in Problem 14 is 20′ 0″ long, what is the safe load?
16. Investigate a steel column whose section is like Fig. 197*B*. It is 20′ 0″ long and carries a load of 250,000 lbs.
17. *a*. What is the safe load on an 8″ × 34 lb. H beam column 15′ 0″ long?
    b. How much would its strength be increased by riveting an 8″ × $\frac{3}{8}$″ plate to each flange?
    c. If one of the above plates only is used?
    d. If a plate 6″ × $\frac{1}{2}$″ is riveted to each side of the web?
18. A working formula for flat end steel columns is given as $P/A = 15{,}000 - 60(L/r)$. The E for this steel is 29,000,000 lbs. per sq. in. What is the limit of validity of the formula?
19. Determine the factor of safety included in the formula for timber columns given in § 185.

CHAPTER XX

ECCENTRIC LOADS AND COMBINED STRESSES

190. Introduction. In the preceding chapters the subject matter has been selected from the simplest possible cases of the particular subject which is being discussed. Thus in dealing with stresses of tension and compression, we have considered only those cases in which the load acts along the axis of the member. Again in the study of beams we have treated only of symmetric bending (§ 131) and further we have made no attempt to discover what might be the *combined* effect of bending and shearing stresses (§ 118 and footnote on page 176).

The subject has been treated in this manner for two reasons: (1) Most of the cases arising in practice are of the simple symmetric type. (2) The principles can be clarified more easily by a study of the less complex cases.

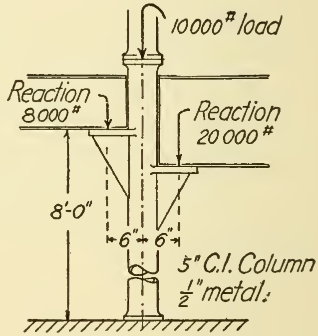


Fig. 323

On the other hand, cases of unsymmetric loadings and combined stresses are by no means rare. Eccentric loading on columns (Fig. 323), roof purlins (Fig. 336), and eccentrically loaded foundations (Fig. 189) are a few cases in point. The treatment of eccentric loading given in this chapter is confined to cases in which the section under stress is symmetric about *at least one* axis. The reason for this lies in the fact that such cases can be treated by use of the ordinary theory of bending (§ 131) only.

If the student does not intend to follow up the general theory of bending (Chapter XXII), the following cases will enable him to solve a large part of the cases that arise in practice. It is well to recognize, however, that these are but special cases of the

general theory, and are treated separately here merely because it can be readily done and for the convenience of those who may wish to stop short of the general case.

Throughout the following discussions the stresses on any cross section will be treated as varying *uniformly*. This is in line with results already quoted in §§ 105 and 130.

191. General Theory of Combined Stresses. The general principle used in the following solutions is that of separating a complex loading into simple component loadings, solving for the unit stresses due to each one, and combining these by algebraic addition. Thus, Fig. 324A shows a bar under axial tension due to the load P and bending due to the load P'.

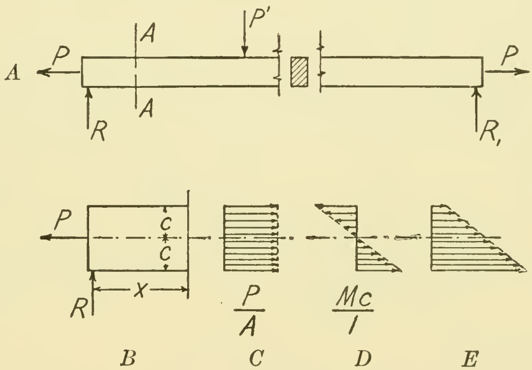


FIG. 324

Let the left-hand part of the bar (section AA) be taken as a free body as in Fig. 324B. The load P causes tensile stresses uniformly distributed over the section whose intensity is P/A, as shown at C. The reaction R causes a bending moment Rx, which is balanced on the cross section by tensile and compressive unit stresses, as shown in D. If we let the moment be represented by M, the amount of the maximum and minimum unit stresses due to this moment are given (§ 135) by the expression $s = Mc/I$.

The separate unit stresses thus determined are combined algebraically in E. That at the top is $(P/A) - (Mc/I)$ and that at the bottom is $(P/A) + (Mc/I)$.

This simple theory is the basis of combined stresses. It is sufficient for most cases, though it can be shown to be in error in the case of members that suffer important deformations. This case is noted in § 197.

The cases that arise divide themselves according to the loading and the type of cross section on which it acts. In the following articles of this chapter, we will discuss various common cases of direct loading and of direct and transverse loading in combination. The sections used in this chapter are all of them symmetric about *at least* one axis. The general case, using a wholly unsymmetric section, is given in Chapter XXII.

192. Eccentric Loading on a Short Block. *Load on one axis of the section.* A solution for one case of eccentric loading is given in § 105. The solution there presented was chosen rather for the purpose of developing the idea of moment of inertia than with reference to this specific problem. That idea having been

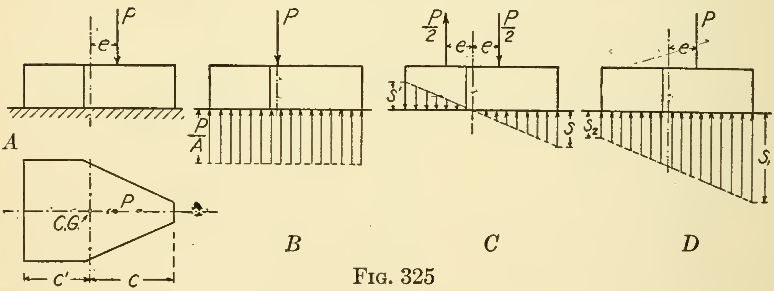


Fig. 325

developed, the following solutions are made to depend on it. Two cases can be distinguished. In the first, the joint between the block itself and its bed is capable of carrying tension. This is the case of a block of masonry bedded on mortar having a definite adhesive strength (Fig. 325), or the case of any part of a bar of continuous material when that part is viewed as a free body (Fig. 327). In the second case, the joint under the block cannot carry tension. This is the case of all foundation blocks and of the stones in an arch built of masonry without appreciable tensile strength (Fig. 326).

A. Tension Possible. Let Fig. 325*A* represent a short block carrying the load P on one axis of the cross section. This load produces in the block two distinct tendencies toward motion: (1) a tendency to move downward (translation), equal to P; and (2) a tendency to rotate (moment), equal to Pe. The first tendency (taken alone) is indicated in Fig. 325*B* and produces uniformly distributed stresses of P/A. The moment Pe, taken by itself, is represented by the couple shown in Fig. 325*C*. This moment would set up a resisting moment as shown. The stresses, determined by § 134, are given by the equations

$$\text{Moment} = Pe = s\frac{I}{c}, \qquad \text{or} \qquad s = \frac{Pec}{I},$$

$$Pe = s'\frac{I}{c'}, \qquad \text{or} \qquad s' = \frac{Pec'}{I}.$$

The actual stresses due to the loading shown in Fig. 325*A* are obtained by adding the component stresses in *B* and *C*. The summation is shown in *D*, where

$$s_1 = \frac{P}{A} + \frac{Pec}{I}, \tag{1}$$

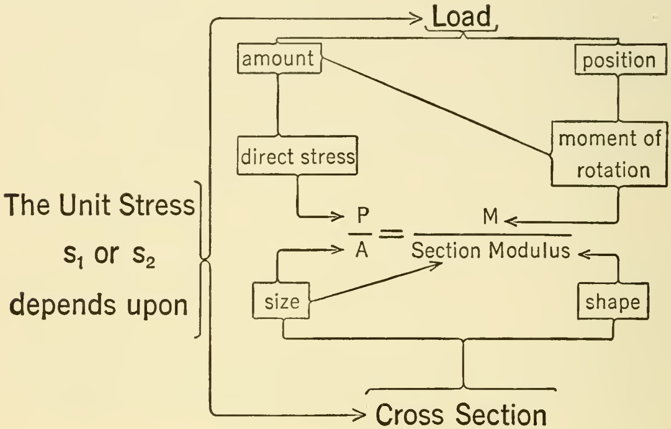
$$s_2 = \frac{P}{A} - \frac{Pec'}{I}. \tag{2}$$

For the case of a section symmetric about two axes, this expression could be written in the form

$$s = \frac{P}{A} \pm \frac{Pec}{I}, \tag{3}$$

in which s represents the unit stress at either edge of the section, depending on whether the $+$ or the $-$ sign is used. Obviously the unit stress caused by an eccentric load will depend on (1) the amount and position of the load, and (2) the size and shape of the cross section on which the load occurs. These physical factors are expressed in equation (3) above. The following diagram is intended to bring out these physical relations more

clearly. From an examination of equation (3) it becomes evident that as e increases s_1 increases and s_2 decreases. Therefore, there is a critical value of $e = I/Ac$ that will cause s_2 to become zero. If e has a value less than this critical value, s_2 is compression. If e is greater than this critical value, s_2 is tension.



For the special case of rectangular section (the most usual case), equations (1) and (2) above reduce to the form given in equations (5) and (6), § 105, viz.,

$$s_1 = \frac{P}{bd}\left(1 + \frac{6e}{d}\right), \tag{4}$$

$$s_2 = \frac{P}{bd}\left(1 - \frac{6e}{d}\right), \tag{5}$$

where d represents the length of the axis of the rectangle on which P occurs and b is the length of the other axis. In equation (5), if $s_2 = 0$, then $e = d/6$. That is to say, if P acts anywhere within the middle third of one axis of a rectangular block, the stresses underneath are compressive throughout the section. This fact is a most important one in the design of masonry structures, constituting the basis of what is known as the "middle-third theory."

B. No Tension Possible. This case will be treated for rectangular sections only. Manifestly it is the same as the

previous case so long as the load lies at the edge of or within the middle third of the axis line; then the conditions of the previous case are unchanged. But when the load lies outside the middle third, if the joint cannot offer tensile resistance, the diagram of stresses becomes like Fig. 326. Here the stress diagram is always a triangle with its center of gravity under the load. The average unit stress is $s/2$, and it is distributed over an area bz. Hence $\Sigma V = 0$ gives rise to the equation

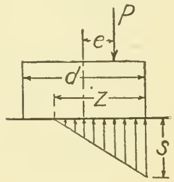


Fig. 326

$$\frac{s}{2}bz = P; \tag{6}$$

and since the center of gravity of the stress triangle lies under the load, we may write

$$\frac{d}{2} - e = \frac{z}{3}. \tag{7}$$

Solving (7) for z, we get

$$z = \frac{3d - 6e}{2}.$$

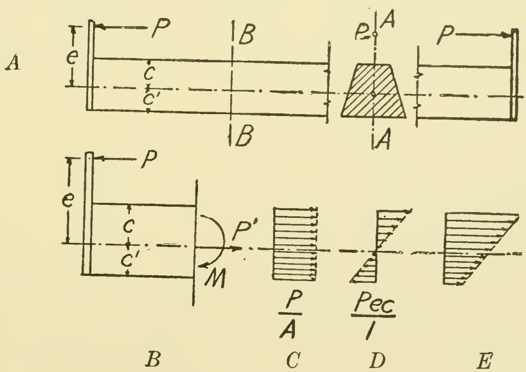


Fig. 327

Substituting this value in (6) and reducing, we find

$$s = \frac{4P}{3bd\left(1 - 2\frac{e}{d}\right)}.$$

Let the student work out the values of s and z for the special positions of the load and interpret the results.

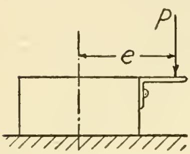


Fig. 328

C. General Remarks. Case A above was treated as for forces p oducing compressive stresses. Precisely the same analysis, merely employing a change of sense throughout, would serve when the forces produce tensile stresses. Fig. 327.

Equation (3), page 289, is valid for all values of e. That is to say, the load P may occur anywhere on the axis, either within the limits of the block or on an extension, as shown in Fig. 328.

193. Eccentric Loading on a Short Block. *Load not on an axis of the section.* This case will be treated only for such sections as are *symmetric about two axes.* The general case is given in Chapter XXII. The effect of the load P, Fig. 329, can be resolved into three components, following the principles of § 192. (1) *The effect of translation.* This is equal to P and, taken by itself, would produce a compressive unit stress of P/A uniformly distributed over the section.

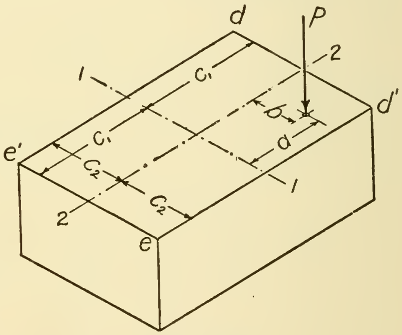


Fig. 329

(2) *A tendency to rotate about the axis 1–1.* This moment is equal to Pa and would produce a uniformly varying unit stress which has a value of $\pm Pac_1/I_1$, at the edges of the block lettered dd' and ee'.

(3) *A tendency to rotate about axis 2–2.* This is equal to Pb and produces a unit stress of $\pm Pbc_2/I_2$, on the edges lettered de' and $d'e$.

The maximum unit stress will evidently occur at the corner lettered d', and will be equal to

$$\frac{P}{A} + \frac{Pac_1}{I_1} + \frac{Pbc_2}{I_2}.$$

The minimum unit stress will occur at e', and will be equal to

$$\frac{P}{A} - \frac{Pac_1}{I_1} - \frac{Pbc_2}{I_2}.$$

The unit stress at d will be equal to

$$\frac{P}{A} + \frac{Pac_1}{I_1} - \frac{Pbc_2}{I_2},$$

and that at e will be equal to

$$\frac{P}{A} - \frac{Pac_1}{I_1} + \frac{Pbc_2}{I_2}.$$

194. Eccentric Loading on Columns. *Load on an axis of symmetry of the section.** If the column is long or the eccentricity of the load is great, so that appreciable deformation from a straight line will occur under loading, the principle given in § 197 will govern. But in most actual cases, the deformations can be neglected. In that event, the unit stresses set up by the load will be the same as in §§ 192A. The allowable unit stresses, however, should be determined from an accepted column formula.

PROBLEMS

1. A right circular cylinder, 6″ in diameter, stands vertically on a base. The joint between the cylinder and the base can carry tensile stress. The top of the cylinder carries a load of 5,000 lbs., concentrated 1″ off the center of the cross section. Determine the greatest and the least unit stress on the joint.
2. Figure 330 shows the plan of a rectangular bearing block. Determine the maximum and minimum unit stress under the block due to a load of 5,000 lbs. concentrated at a. The joint between the block and its base can resist tensile stress.

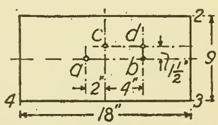


FIG. 330

3. In Problem 2, let the load be at b.
4. Solve Problems 2 and 3, assuming that the joint cannot offer tensile resistance. In each case determine the line of zero stress on the joint.

* For the case of an unsymmetric section, see § 232A.

5. Let the block in Problem 2 carry a load of 6,000 lbs. at c. (*a*) Let the joint be able to carry tension, (*b*) let the joint be unable to carry tension.
6. Let the block in Problem 2 carry a load of 6,000 lbs. at d and let the joint be able to carry tension. (*a*) Determine the unit stress at each of the four corners of the block. (*b*) Determine the line of zero stress on the joint.
7. A rectangular block 6″ × 16″ in plan carries two loads applied in the middle of the width of the top face. The load P is 2″ from the center while P_1 is 3″ from the center. Compute the ratio between P and P_1 so that the unit stress on the base may be uniform.
8. A tensile load of 50,000 lbs. is applied to the web of a 6″ × 12¼ lb. I beam. The point of application of the load is 1¼″ above the center of the web. What is the maximum and the minimum unit stress?
9. (*a*) Where must the load in Problem 8 be applied so that the minimum stress on an outermost fiber is zero? (*b*) What, then, is the maximum unit stress on any part of the section?
10. What are the maximum and minimum unit pressures under the foundation shown in Fig. 189?
11. Investigate the column in Fig. 331.
12. Investigate the column in Fig. 323.
13. Investigate the column in Fig. 332*A*.

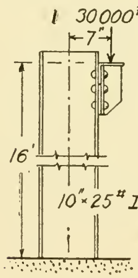


FIG. 331

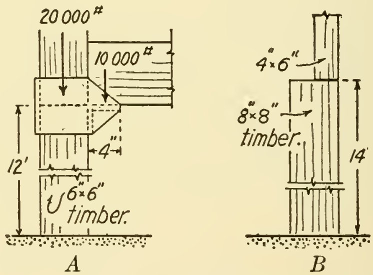


FIG. 332

14. What load can the column in Fig. 332*B* safely receive from the 4″ × 6″ post above it?
15. A cast iron column, 10″ in diameter outside and 8″ in diameter inside, carries three loads of 12,000 lbs. each. Each load is 12″ from the center of the column. On plan one of the loads is separated from each of the other two by 90° intervals. Is this a safe loading?

195. Transverse Loading. *Not in an axial plane of the section.* We will consider here only the case of loading that lies in a plane intersecting the longitudinal axis of the beam and of sections that have *at least* one axis of symmetry.*

* For the case of an unsymmetric section, see § 232B.

Let Fig. 333*A* represent a beam loaded at its center with the load P. This load can be reduced to two component loads in the planes of the principal axes of the section, represented by P_1 and P_2, Fig. 333*B*. Then the unit stress along the edges ab and $a'b'$, due to P, will be found from the equation

$$\frac{P_1L}{4} = \frac{s_1I_1}{c_1}, \quad \text{or} \quad s_1 = \frac{P_1L}{4}\frac{c_1}{I_1},$$

and the unit stress along the edges aa' and bb' will be

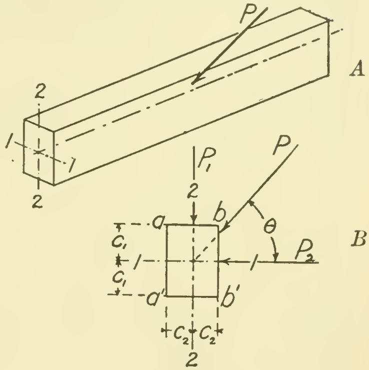
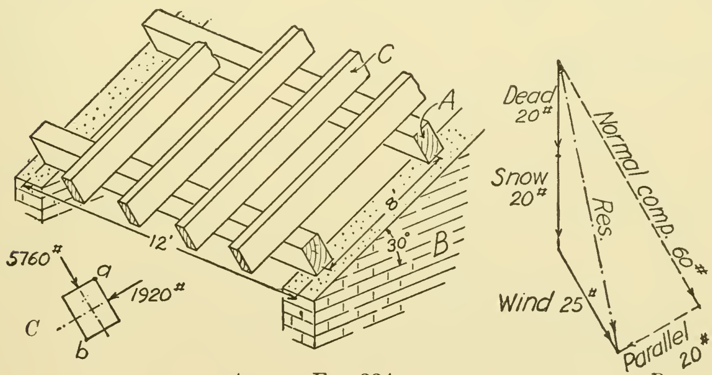


FIG. 333

$$\frac{P_2L}{4} = \frac{s_2I_2}{c_2}, \quad \text{or} \quad s_2 = \frac{P_2L}{4}\frac{c_2}{I_2}.$$

The maximum and minimum stresses are found by combination, as in § 191.

196. Purlins. A good illustration of the principle of § 194 is furnished by a roof purlin. The I beam A in Fig. 334 illustrates the function of a purlin. It covers the span between main



A FIG. 334 B

supports of a roof (trusses in this case) and runs parallel to the eaves. It carries other members which run transversely to it.

As a definite case, consider the construction shown in Fig. 334*A*. The roof is assumed to extend indefinitely above and below the part shown in the figure. The members *A* are purlins, supported on the walls *B* and carrying the rafters *C*. Let the weight of the roof construction be 20 lbs. per sq. ft., the possible snow load 20 lbs. per sq. ft., and the wind load (which is taken as a rule as being perpendicular to the roof) 25 lbs. per sq. ft. These loads are shown and resolved into components perpendicular and parallel to the roof plane in Fig. 334*B*.

Let the purlin be 8″ × 10″ in cross section, as shown in Fig. 334*C*. Each purlin will carry 12 × 8 = 96 sq. ft. of roof and its component loadings will then be 96 × 60 = 5,760 lbs. (perpendicular) and 96 × 20 = 1,920 lbs. (parallel), as shown in Fig. 334*C*. Each of these is uniformly distributed over the length of the purlin. The bending moments are

Perpendicular:

$$\frac{5{,}760 \times 12 \times 12}{8} = 103{,}700 \quad \text{lbs. ins.}$$

Parallel:

$$\frac{1{,}920 \times 12 \times 12}{8} = 34{,}560 \quad \text{lbs. ins.}$$

The unit stresses are found from the equations

Perpendicular:

$$103{,}680 = s\,\frac{\dfrac{8(10)^3}{12}}{\dfrac{10}{2}}; \qquad s = 777 \ \text{lbs. per sq. in.}$$

Parallel:

$$34{,}560 = s'\,\frac{\dfrac{10(8)^3}{12}}{\dfrac{8}{2}}; \qquad s' = 324 \ \text{lbs. per sq. in.}$$

The maximum bending unit stress occurs on the corner *a* in compression, or *b* in tension, and is equal to 1,100 lbs. per sq. in.

PROBLEMS

1. An 8″ × 12″ timber, 14′ 0″ long, is set with its 12″ sides making an angle of 30° with the vertical. It carries a uniformly distributed load of 4,000 lbs. which acts vertically. Determine (1) the maximum unit stresses in bending, (2) the line of zero stress on the cross section at the center of the length, (3) the line of zero stress on the cross section 4′ 0″ from one end of the timber.

2. A 6″ × 12¼ lb. I beam, 8′ 0″ long, is placed with its web vertical. It carries a vertical concentrated load of 1,000 lbs. at the center of the span and a uniformly distributed horizontal thrust of 100 lbs. per foot of its length. Draw a diagram showing how the bending stresses vary along the length of the most stressed fibers and determine the maximum bending unit stress in the beam.

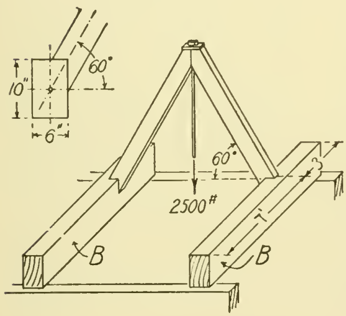


Fig. 335

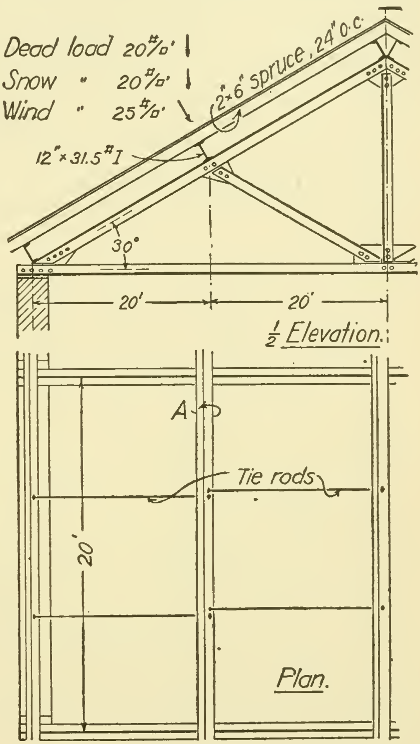


Fig. 336

3. What is the safe load on the timber in Problem 1?

4. Two 8″ × 18 lb. I beams placed side by side project from a building as horizontal cantilevers 6′ 0″ long. The webs are vertical and 8″ apart. A wedge block 9″ wide is driven between the webs at the outer end of the cantilever and a load of 3,500 lbs. is suspended from it. Investigate the safety of the beams.

5. Investigate the stresses in the purlins in Fig. 336. The tie rods cause the purlin to become a continuous beam of three spans *as against forces parallel to* the roof plane.

6. Investigate the stresses in the beams *B*, Fig. 335.

197. Bending and Direct Stress. By direct stress we mean stress acting directly along the member; tension or compression. Not infrequently direct stress occurs in a member which also is subjected to bending. Thus the pole in Fig. 93 is under both bending and compression due to the vertical loads and also due to the pull on the back-stay. A similar action occurs in truss members which carry tension or compression due to the external loading and bending due to their own weight.

Figure 337A may be taken as a diagrammatic representation of a simple case of bending and direct stress. Let the left-hand end of the member be cut loose, as shown in Fig. 337B. In order to maintain equilibrium, we must find on the cut section (a) a shear S, equal to R; (b) a direct stress P', equal to P; and (c) a resisting moment M, equal to Rx. The shear may be treated separately as in the case of ordinary bending, but the stress due to P' and that due to M, having the same directions, can be combined just as in §§ 190–195. In general, there are three solutions, depending on whether the deflection due to bending is considered or not (§ 194).

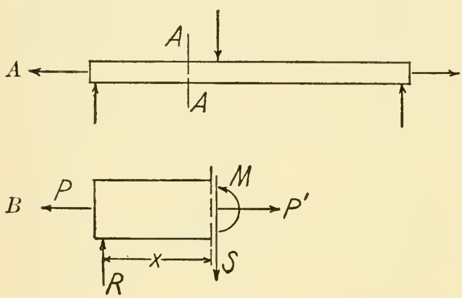


Fig. 337

(1) *Deflection neglected.* The maximum and minimum unit stresses can be determined by simple combination, just as in §§ 192–193.

(2) *Deflection considered.* In Fig. 338A is shown an eye bar carrying the direct load P and under bending due to the loads P' and P''. The deflection y due to bending is shown greatly exaggerated. Because of this deflection, there is a (negative) bending moment of $-Py$ in addition to the moment M due to the weight of the bar and the applied loads.

The unit stresses due to the various loadings are shown in Fig.

338. In C is the uniform distribution of stress due to direct tensile loading and equal to P/A. In D the maximum unit stress is Mc/I due to the weight of the bar and the loads P' and P''. In E are shown the stresses due to the force P acting with the lever arm y. The maximum is Pyc/I. The maximum unit stress may be either at the top or bottom fiber and is obtained by algebraic addition.

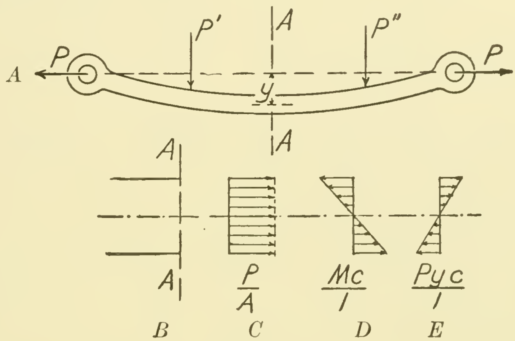


Fig. 338

In working out actual values for this case, it is important to choose the section AA so that the values obtained are the maximum possible. Also, the determination of the deflection y leads to the following two possible solutions.

(*a*) *An approximate solution.* In this case the deflection y is made to depend on the weight of the bar and the applied loads only. It can, therefore, be determined quite easily in most cases from the values given in Table III. The examples worked out in § 198 are treated in this manner.

(*b*) *An exact solution.* The solution outlined in (*a*) above neglects the fact that the actual deflection y depends not only on the transverse loading, but also on the moment $-Py$. Such a solution can be worked out by means of the general equation of the elastic curve (§ 152), but this will not be attempted here. It can be found in more extended texts. The actual differences between the approximate and the exact solutions will naturally depend on the particular case. The stiffer the member and the smaller the loadings, the less will be the differences.

198. Illustrative Problems. A steel eyebar $2'' \times 10''$ and $25'\ 0''$ long, similar to Fig. 338, carries an axial tensile load of 200,000 lbs. Let it be required to find the maximum and minimum unit stresses due to the axial load and the weight of the bar. The $10''$ side is vertical.

(*a*) *Deflection neglected.* The weight of the eyebar is

$$\frac{2 \times 10}{144} \times 25 \times 490 = 1{,}701 \qquad \text{lbs.}$$

The B. M. due to the weight is

$$\frac{1}{8} \times 1{,}700 \times (25 \times 12) = 63{,}790 \qquad \text{lbs. ins.}$$

The unit stress due to this moment is

$$\frac{63{,}790 \times 12 \times 5}{2 \times 10 \times 10 \times 10} = 1{,}914 \qquad \text{lbs. per sq. in.}$$

The unit stress due to direct load is

$$200{,}000 \div (2 \times 10) = 10{,}000 \qquad \text{lbs. per sq. in.}$$

The maximum and minimum unit stresses will occur at the center section where the bending is greatest. They will be:

on the bottom fiber: $10{,}000 + 1{,}914 = 11{,}914$ lbs. per sq. in.

on the top fiber: $10{,}000 - 1{,}914 = 8{,}086$ lbs. per sq. in.

(or, within limits of the approximation, 11,920 on bottom fibre and 8,090 on top fibre).

(*b*) *Deflection considered* (*approximate solution*). Deflection due to the weight of the bar (assuming $10''$ side vertical) is

$$\frac{5}{384} \times 1{,}700 \times (300)^3 \div \left(29{,}000{,}000 \times \frac{2 \times (10)^3}{12}\right) = 0.124''.$$

The moment due to the weight of the bar is (see above)	63,790	lbs. ins.
The moment due to the direct load is; $200{,}000 \times 0.124 =$	24,800	
The total moment is	38,990	lbs. ins.

The unit stress due to the total moment is

$$\frac{38{,}990 \times 12 \times 5}{2 \times 10 \times 10 \times 10} = 1{,}170 \quad \text{lbs. per sq. in.}$$

The unit stress due to direct load is: 10,000 lbs. per sq. in. (see above).

Thus the maximum and minimum stresses will be

on the bottom fiber: $10{,}000 + 1{,}170 = 11{,}170$ lbs. per sq. in.
on the top fiber: $10{,}000 - 1{,}170 = 8{,}830$ lbs. per sq. in.

(or, 11,170 and 8,830 respectively for the bottom and top fibres)

(*c*) *Exact solution.* By the exact solution, the unit stress due to the moment would be found to be 1,134 lbs. per sq. in. and the maximum and minimum unit stresses would be determined as 11,134 lbs. per sq. in. and 8,866 lbs. per sq. in.

In the above examples, the more exact solutions show the smaller unit stress. If the axial load had been compressive, the exact solutions would have given the larger stresses.

(*d*) *Compression and bending.* When axial compression is combined with bending, the solution is exactly like the preceding ones except that the stresses combine differently, due to the difference in sense of the axial force.

If the member is long enough to class as a column, the allowable unit stress should be determined from a column formula.

199. Rafters. The principle of § 198 is useful in solving for the stresses in an ordinary rafter. In Fig. 339*A* is shown a pair of rafters, each of which carries a concentrated load of W lbs. at its center. Below is shown one of the pair of rafters as a free body. The vertical component of the reaction is W. The joint between the rafters is assumed to be frictionless. Then the reaction from the adjoining rafter will be horizontal. Let it be R. Then the H component of the wall reaction is the same. Taking a center of moments at o, the positive moment of the load is $WL/2$. The negative moment of R is $RL \tan \phi$. Then

$$\frac{WL}{2} = RL \tan \phi, \qquad \text{and} \qquad R = \frac{W}{2 \tan \phi}.$$

In Fig. 339B are shown also the load and the reactions resolved into components perpendicular and parallel to the rafter. It can

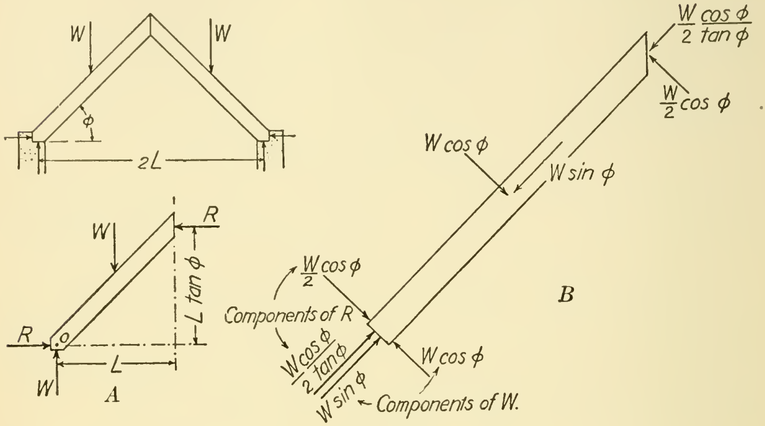


FIG. 339

now be seen that the rafter is subjected to bending due to a central load of $W \cos \phi$, on a span of $L/\cos \phi$. The bending moment of this load is

$$\frac{W \cos \phi \left(\dfrac{L}{\cos \phi}\right)}{4} \quad \text{or} \quad \frac{WL}{4}.$$

If the rafter has a rectangular cross section and the breadth is b and depth d, then the stress due to bending is found from the equation

$$\frac{WL}{4} = s\frac{I}{c} = s\frac{bd^2}{6},$$

whence we have

$$s = \frac{3WL}{2bd^2}. \tag{8}$$

The compressive stress, due to direct loading, in the top half of the rafter (the part above the load) is less than in the bottom half, the latter being

$$\frac{W}{2}\left(2 \sin \phi + \frac{\cos \phi}{\tan \phi}\right);$$

and the compressive unit stress is

$$(9) \qquad \frac{W}{2bd}\left(2 \sin \phi + \frac{\cos \phi}{\tan \phi}\right).$$

The maximum unit stress would then be found by adding (8) and (9) above.

It will be noted that in (9) above the term $\cos \phi/\tan \phi$ (and hence the whole expression) is very large for small values of ϕ, also that the whole expression becomes W/bd when $\phi = 90°$. For the usual slopes of roofs, the whole quantity will vary between $3W/2bd$ at about 25° to about W/bd at 60°. It will also be noted that the bending stress does not depend upon the slope, but is the same as it would be in a simple beam of span L.

To make the above equations applicable to beams of sections other than rectangular, substitute I/c for $bd^2/6$ in (8) and substitute area for bd in (9). Sometimes a rafter is fastened to purlins, as shown in Fig. 335. In that case, the directions of the reactions will be determined by the way in which the fastening is made, whether at one or both ends, but the principles of combined stress will serve for the solution.

PROBLEMS

1. If the beam in Fig. 340 is a 3″ × 5.5 lb. I beam and the 800 lb. forces are applied axially, what are the maximum unit stresses as determined by each of the methods (*a*) and (*b*) of § 198?
2. Repeat Problem 1, changing the 800 lb. forces to compression.
3. If the beam in Fig. 340 is a 6″ × 6″ timber, how much uniformly distributed load can be added before the safe load on the beam is reached? Use approximate method.

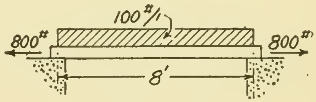


Fig. 340

4. In Problem 1, what would be the maximum unit stress if the tensile forces were applied at the top of the beam?
5. In Fig. 340, where should the tensile forces be applied in order to make the maximum stress in the beam as small as possible?
6. In Fig. 339, let W = 500 lbs., ϕ = 30° and L = 10′ 0″. Let the rafters be 2″ × 10″ timbers. Determlne the maximum unit stress on the rafters and the thrust on the walls.
7. Investigate the stresses in the rafters in Fig. 336.
8. Investigate the stresses in the rafters C, Fig. 334. Let the rafters be $1\frac{1}{2}$″ wide, 5″ deep, and spaced 2′ 6″ apart. The live and dead loads are to be taken as in § 196. Let the rafters be fastened at the top end only.

200. Direct Stress and Shear. In §§ 74 and 75 it was shown that forces which produce tensile or compressive *stresses* in a body also produce *shearing stresses*, and vice versa. But in those cases the forces were simple tensile, compressive, or shearing forces. We will now discuss the case where both tensile and shearing *forces* are present.

Let Fig. 341A represent such a case, and let Fig. 341B represent a small parallelepiped cut from the block and having the dimensions a, b, and c as shown and a thickness (perpendicular to the

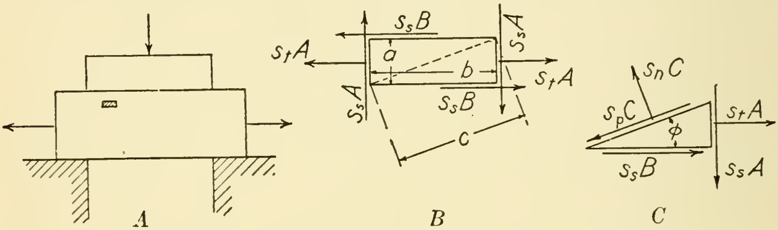


FIG. 341

paper) of unity. Also let s_t be the *unit* tensile stress on the vertical faces and s_s be the *unit* shearing stress on either of the vertical or horizontal faces (§ 73). Then the *total* forces acting on the various faces of the block are s_tA, s_sA, and s_sB, as shown.

Now let the block be cut by a diagonal plane and one half of it shown free, as in Fig. 341C. It is under the action of three forces, s_tA, s_sA, and s_sB. On the cut section the stresses may be represented by a parallel stress whose intensity is s_p, and a normal stress whose intensity is s_n. The total stresses then are s_pC and s_nC, as shown. By resolving the three external forces into components perpendicular and parallel to the diagonal, we will get the amount of s_nC. Doing this, we find

$$s_nC = s_tA \sin\phi + s_sA \cos\phi + s_sB \sin\phi,$$

$$s_nC = s_tC \sin^2\phi + s_sC \cos\phi \sin\phi + s_sC \cos\phi \sin\phi,$$

$$s_n = s_t \sin^2\phi + 2s_s \cos\phi \sin\phi$$

$$= s_t \sin^2\phi + s_s \sin 2\phi,$$

or

$$(10) \qquad s_n = \tfrac{1}{2}s_t(1 - \cos 2\phi) + s_s \sin 2\phi.$$

By a similar process, it can be shown that

(11) $$s_p = \tfrac{1}{2}s_t \sin 2\phi + s_s \cos 2\phi.$$

From these expressions, we see that if we choose our elementary parallelepiped in one proportion (thus giving ϕ a definite value) we get one set of values. Another proportion gives a different set of values. This evidently means that the stresses set up by direct and shearing forces are different on planes cutting the piece at different angles. There is one plane on which these stresses have maximum values. By differentiating equation (10) with respect to ϕ and placing the derivative equal to zero, we find that s_n is a maximum when

(12) $$\tan 2\phi = -\frac{2s_s}{s_t}\cdot$$

Substituting this value of ϕ in equation (10), we get

(13) $$\text{maximum value for } s_n = \tfrac{1}{2}s_t \pm \sqrt{\left(\frac{s_t}{2}\right)^2 + s_s^2}.$$

By a similar process it can be shown that s_p is a maximum when

$$\tan 2\phi = \frac{s_t}{2s_s},$$

and

(14) $$\text{maximum value for } s_p = \sqrt{\left(\frac{s_t}{2}\right)^2 + s_s^2}.$$

In equation (13) the minus sign is to be used when the direct force is compressive.

In equations 10–14 above, if we let s_t approach zero we are, in effect, changing our case to that of shear, uncombined with direct stress. If this is done, equation (12) gives $\phi = 45°$ and equation (13) gives $s_n(\text{max}) = s_s$. This is equivalent to saying that when no tensile or compressive forces are acting the maximum normal stresses occur on planes at 45° to the horizontal and are equal in intensity to the shearing stresses as already shown in §§ 74 and 75.

PROBLEMS

1. Within a solid object there exists a horizontal tensile stress of 750 lbs. per sq. in. and vertical and horizontal shearing stresses of 500 lbs. per sq. in. Find the direction and magnitude of the maximum shearing stresses.
2. In Problem 1, find the unit shearing stress on planes making 20° and 30° with the horizontal.
3. What is the direction and the amount of the maximum tensile stress in Problem 1?
4. A bolt $\frac{3}{4}''$ in diameter carries a tensile load of 2,000 lbs. and a shear of 3,000 lbs. Find the direction and intensities of the maximum tensile and shearing stresses.

201. Combined Stresses in Rectangular Beams. The unit stress due to bending which occurs in any part of a beam can be determined by use of equation (7), § 134. The unit shearing stress can be determined by equation (3), § 140. Heretofore, we have treated these stresses separately, although they occur simultaneously and actually produce resultant stresses which are different in amount and direction from the component bending and shearing stresses. These differences, under certain circumstances, may become important in determining the safety of a given beam.

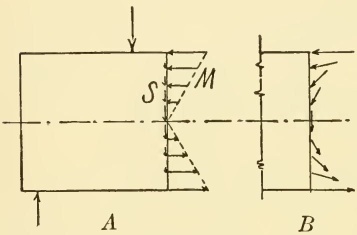


Fig. 342

In Fig. 342*A* is shown the left end of a loaded beam. The bending moment stresses (as heretofore determined) are shown at *M*. The vertical shearing stresses are indicated at *S*. It should be remembered (though the drawing cannot show this fact) that the intensity of the shearing stress is greatest where the intensity of the bending stress is least and vice versa. Now evidently the *resultant* stresses on the section will vary from horizontal at the outermost fibers to vertical at the neutral surface, as shown diagrammatically in Fig. 342*B*. It will be shown later that the horizontal shearing stresses also influence the resultants, but the principle that the resultant stresses may vary in amount and in direction, from those already studied, is made clear by the figure.

The principle of § 200 and the equations there developed furnish the basis for determining the variations in resultant stress, but the complete solution of a definite case is necessarily a

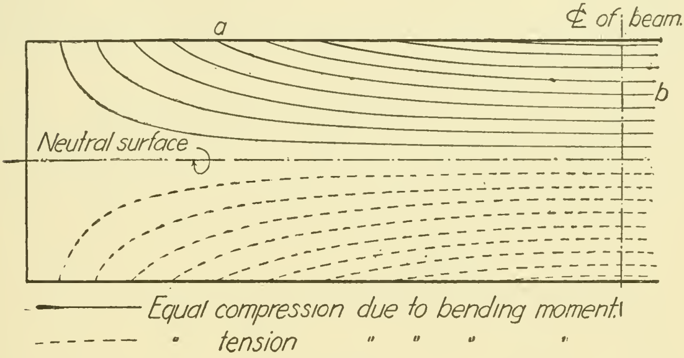


Fig. 343

long and tedious process. The purpose of this article is not especially to follow through some definite case in detail, but to furnish a visual concept of the variations which occur and to draw some conclusions which will be helpful in actual designs.

For this purpose Figs. 343, 344, 345 have been prepared. They are all based on a rectangular beam $2'' \times 20''$ in section, and $100''$ long, carrying a uniformly distributed load of 100 lbs.

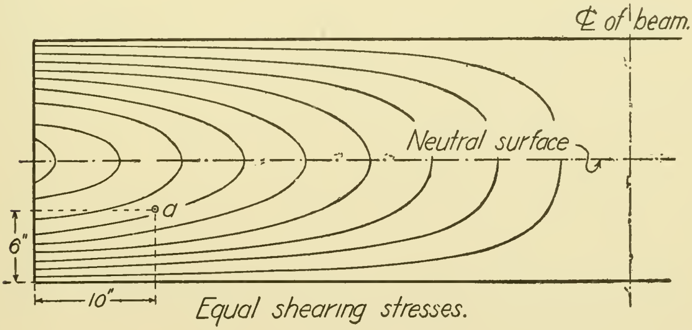


Fig. 344

per inch. Figure 343 shows the variation in bending moment stresses. The curves drawn on the beam are lines of equal stress; i.e., the line marked *ab* passes through all points at which

the bending moment stresses are 500 lbs. per sq. in. in compression. Figure 344 is a similar diagram showing lines of equal unit shearing stress. It should be remembered that in the case of shear there is a vertical and a horizontal shear of equal intensity (§§ 73 and 139). The diagram expresses either the *H* or the *V* unit shearing stress.

We can now proceed to determine resultant stresses. Let the point marked "*a*" be chosen. At this point the tensile unit stress due to bending is 135 lbs. per sq. in., and the unit stress due to horizontal (or vertical) shear is 126 lbs. per sq. in.

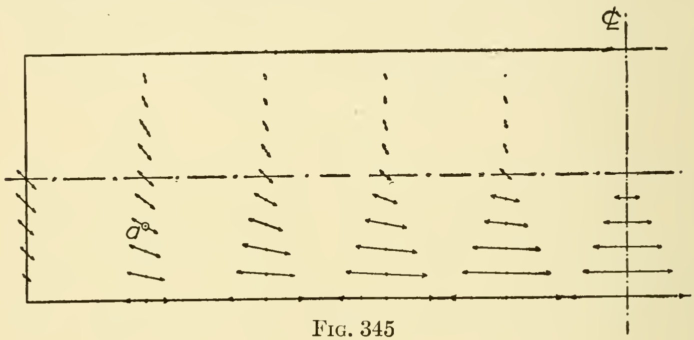


Fig. 345

By equation (12), § 200, the inclination of the maximum resultant stress is given by the equation $\tan 2\phi = -(2 \times 126)/135 = -1.88$, which gives $\phi = -31°$ or $59°$. The plane inclined at 59° is the one on which the resultant tension occurs while on the other is found the maximum compression. Then from equation (13), the maximum unit tensile stress is

$$s_n\,(\text{max.}) = \tfrac{1}{2}(135) + \sqrt{\left(\frac{135}{2}\right)^2 + (126)^2} = 210 \qquad \text{lbs. per sq. in.}$$

The unit compressive stress at this same point, found by using the minus sign in equation (13), is 75 lbs. per sq. in. and is inclined at 90° to the tensile stress.

A series of such computations has been made for various points on the beam shown in Figs. 343 and 344. The results are shown in Fig. 345. Here the resultant stresses are shown by the

lengths and inclinations of the arrows. Only the resultant tensile stresses in the left-hand half of the beam are shown. By turning the diagram upside down the compressive stresses in the right-hand half will appear. From this diagram the way the resultant tensile stresses change in amount and direction can

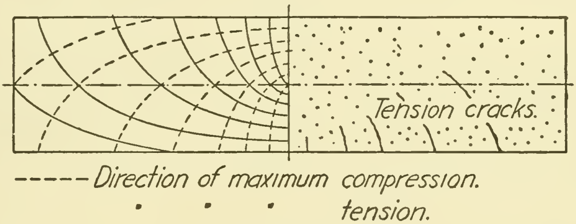


FIG. 346

easily be seen. Figure 346 has been drawn to show the full beam and the variation *in direction only* of the resultant tensile and compressive stresses. The curves in this figure form two sets which, at a point of intersection, are at right angles to each other. This agrees with the critical values of ϕ found in connection with equations (12) and (13) of § 200.

Diagonal stresses of tension, compression, and shear occur in all beams, as shown above. Ordinarily they are not important, since the horizontal stresses determined in the usual manner (§§ 134 and 140) are actually the maximum values. But in certain special cases the diagonal stresses become important.

(1) *In the case of beams with thin webs* (see § 204).

(2) *In the case of reinforced-concrete beams.* In testing such beams, cracks frequently develop as shown in Fig. 346. These result from the diagonal tensile stresses and, in a general way, the cracks run normal to the lines of stress. The placing of reinforcing rods near the ends of the beam is vitally affected by the diagonal tension.

(3) *In the case of wooden beams.* In tables giving the strength of wood, it is not unusual to see the longitudinal shearing strength of *beams* listed at a smaller value than the strength in shear parallel to the grain, for the same material. This is due partly to the serious effect of the usual defects in the timber, such as

shakes and checks, and partly to the fact that wood is very weak in tension across the grain, and hence the diagonal tension is very apt to be a serious factor, particularly if the timber is cross grained. These complex factors are all expressed in a lowered working unit stress.

PROBLEMS

1. A wooden beam is 10″ × 12″ and 10′ 0″ long. It carries a uniformly distributed load of 10,000 lbs. Find the amount and direction of the maximum unit stresses in tension, compression, and shear at points which are taken on sections 2′ 0″ from the end and 2′ 0″ from the center and at distances of 4″ and 2″ from the neutral surface.

202. Beams with Thin Webs. Plate girders and I beams are shaped specially to resist bending stresses with a minimum expenditure of metal. The shapes in use vary in detail, but in general they provide large concentrations of metal in the flanges (to resist bending moments) and deep thin webs which connect the flanges and transmit the shear. Typical shapes are shown in Figs. 254 and 256.

It is obviously impossible to make any shape that will be equally economical whether called upon to resist large bending moments and small shears (as in a beam with long span and relatively light load) or to resist smaller bending moments and large shears (as in a short beam with a relatively heavy load). The I beam shapes in common use are compromises intended to meet the usual run of conditions. Therefore, in extreme cases of span or load, special attention must be given to the actual maximum stresses that may arise through the combined effects of bending, shear, and the localization of stress due to concentrated loadings on the flanges or web.

In general, it may be said that failures of beams with thin webs occur in one of the following ways:

(1) For beams of *ordinary spans.* Failure comes by a gradual sagging. For this case the ordinary theory of bending (§ 134) furnishes a satisfactory solution.

(2) For beams of *long spans.* The top flange of the beam, being in compression, acts somewhat like a column and the

beam may fail by a sidewise buckling of the top flange. (See § 203B.)

(3) For *short spans.* Failure usually is due to shearing, either as the direct result of excessive shearing stresses or indirectly through a combination of shearing and compressive stresses. (See § 204B.)

(4) By *localized stresses.* Where heavy concentrations of loading occur. (See §§ 204C, 204D.)

While the usual theory of bending is well supported by experimental evidence, the cases outlined in 2, 3, and 4, above, have not received the attention they deserve.* Moreover, the combined stresses involved are necessarily very complex. As a result, the treatment of these cases involves many assumptions and the formulas in use are very largely empirical.

NOTE. In the following articles, the commonly used assumptions and the resulting formulas will be discussed in a general and very condensed fashion.* For the sake of shortening the discussions, a simplified beam section (Fig. 347) will be used. This section omits the curved fillets and sloped flanges usual in I beam sections, but retains the heavy flange and light web section typical of all beams in the class under discussion.

203. Flange Stresses on Beams with Thin Webs.* A. DISTRIBUTION OF RESISTING MOMENT. The theory of bending given in Chapter XIV shows how the bending stresses are distributed on any beam section. The equations there derived hold good for beams with thin webs. But for beams with very large sections it is sometimes desirable to use a shorter, approximate method in computations. This is based on the fact that the flanges of such a beam furnish by far the greater part of the resisting moment.

Figure 348 shows the distribution of bending stress on a beam with a cross section like Fig. 347. The *unit* stresses on the *flange* vary from s to s_1 and are distributed over a large area. The unit stresses on the *web* vary from s_1 to zero and the average value is

* The most important tests available are published in the BULLETIN OF THE UNIVERSITY OF ILLINOIS ENGINEERING EXPERIMENT STATION, Nos. 68 and 86. A very good general discussion of these phenomena is given in Hool & Kinne, *Structural Members and Connections.*

s_2, being distributed over the thin web section. Moreover the flange stresses have a much larger lever arm from the neutral surface than the web stresses. It thus becomes evident that the flange stresses make up by far the larger part of the resisting moment. The approximate rule (quoted later) used in figuring the resisting moment of plate girders is based on this fact.

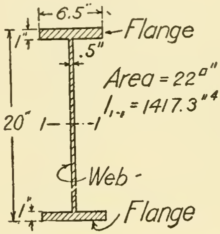


FIG. 347

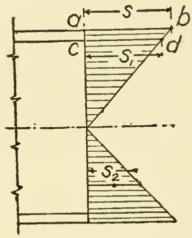


FIG. 348

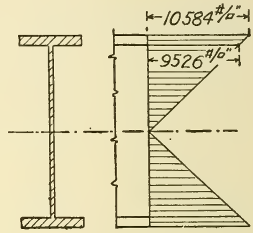


FIG. 349

To illustrate the point let us take a beam whose section is like Fig. 347, loaded with 100,000 lbs., uniformly distributed on a span of 10′ 0″. The maximum bending moment is 1,500,000 lbs. ins. and the unit stress in the outermost fiber of the central section is found from

$$(15)\quad 1{,}500{,}000 = s\,\frac{1417.3}{10}; \qquad s = 10{,}584 \quad \text{lbs. per sq. in.}$$

The unit stress at the joining of the flange and the web is

$$10{,}584 \times \frac{9}{10} = 9{,}526 \qquad \text{lbs. per sq. in.}$$

These unit stresses are shown at scale in Fig. 349. The total stress on one flange is

$$\tfrac{1}{2}(10{,}584 + 9{,}526) \times 6\tfrac{1}{2} \times 1 = 65{,}358 \qquad \text{lbs.}$$

The total stress on one half of the web is

$$(9{,}526 \div 2) \times (9 \times 0.5) = 21{,}433 \qquad \text{lbs.}$$

The resisting moment of the flange stress is found by multiplying the total stress in one flange by the distance from the neutral surface to the center of gravity *of the stresses*, and then doubling

the result to account for both flanges. In this case the center of gravity of the flange *stresses* is at 9.5087″ from the neutral surface. The resisting moment of the flange stress is then

$$65{,}358 \times 9.5087 \times 2 = 1{,}243{,}000 \qquad \text{lbs. ins.}$$

and the resisting moment of the web stress is

$$21{,}433 \times 6 \times 2 = 257{,}100 \qquad \text{lbs. ins.}$$

In this case, the web actually contributes about one-fifth of the entire resisting moment.

In plate girders, the flanges are proportionately heavier and they carry a larger percentage of the moment stresses. It is a common rule in designing such girders to assume that the flanges take all of the moment and that the moment stresses are distributed uniformly over the flanges. This rule, applied to the beam in question, would give a distribution of stress such as is shown in Fig. 350. Here the resultant stresses in the flanges are shown by the arrows C and T and the distance between the *centers of gravity* of the flanges is L (sometimes called the lever arm of the stress couple). The resisting moment is either CL or TL (since $C = T$). Using the same case as above and solving for the unit stress on this basis,

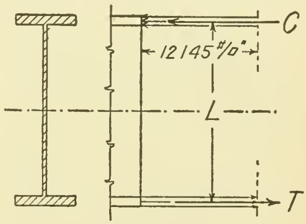


FIG. 350

$$1{,}500{,}000 = 19C; \qquad C = 78{,}950 \qquad \text{lbs.,}$$
$$78{,}950 \div (6.5 \times 1) = 12{,}150 \qquad \text{lbs. per sq. in.,}$$

which is the unit stress on the outermost fiber. This is about 20 per cent. in excess of the exact solution (given by equation (15)), but in girders with heavier flanges the discrepancy is less.

B. FOR LONG SPANS. The top flange of a beam is under compressive stresses which vary in intensity with the bending moment. When the beam is long, these stresses cause the flange to act somewhat as a column. If the flange has no sidewise

support, it is apt to buckle and cause the failure of the beam. Where effective lateral support is provided at intervals, the distance between supports becomes the critical factor, as in a braced column.

The column action set up in the flange is not the same of course as in a free standing column. The web restrains the flange in a vertical direction, and the loading occurs throughout the length, causing stresses which vary throughout the length. Therefore, the ordinary column formulas cannot be applied directly, but are modified according to the judgment of the designer and to correspond with whatever test results are available.

The general principle followed in all formulas used to control flange buckling is, however, the same as that used in column formulas. The unit stress allowable for short pieces is reduced by a factor which depends on the width of the flange.

The formula given in the specifications of the American Bridge Company is as follows:

$$s_c = 19{,}000 - 300\frac{L}{b},$$

where s_c is the maximum allowable compressive unit stress in the top flange of a beam in pounds per square inch, L is the unsupported flange length in inches, and b is the width of the flange in inches. This formula is restricted to values of L/b between 10 and 40. It is evidently based on the column formula used in the same specifications, viz., $s_c = 19{,}000 - 100\ L/r$ (p. 280). In transforming this formula, the flange is supposed to have a radius of gyration of $\frac{1}{3}$ of its width, which is approximately what it would be if the flange were a rectangle. This formula makes reductions which are rather more severe than others in common use.

PROBLEMS

1. The plate girder section shown in Fig. 241 is used to carry a uniformly distributed load of 130,000 lbs. on a span of 30′ 0″. What is the maximum bending unit stress as figured by the approximate method of § 203A and by the exact method?

2. If the approximate method of § 203A be applied to figuring the resisting moment of a 24″ × 80 lb. I beam, what percentage of error will be involved?
3. What is the safe uniformly distributed load on a 12″ × 31½ lb. I beam which is 15′ 0″ long and without lateral support for the top flange?
4. What size I beam is required to carry a uniformly distributed load of 6,000 lbs. on a span of 14′ 0″, the top flange being unsupported?
5. What is the relative capacity of an 8″ × 34 lb. H beam as against two 8″ × 17½ lb. I beams in carrying a uniformly distributed load on a span of 14′ 0″, the top flange being unsupported?
6. If the plate girder in Problem 1 is unsupported laterally, what is the safe load concentrated at the center?

204. Web Stresses in Beams with Thin Webs.* A. General Considerations. The beam shown in Fig. 347, loaded with 100,000 lbs., uniformly distributed on a span of 10′ 0″, will be used as a basis of discussion. According to the general theory of bending, the shearing unit stresses will be distributed over the section of greatest shear, as shown in Fig. 351. Let the student check the indicated unit stresses.

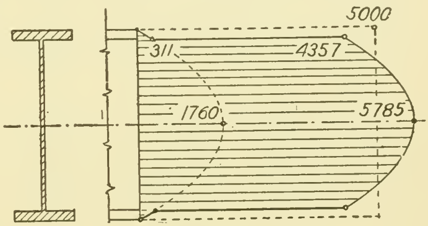


Fig. 351

It is evident from the diagram that the web carries by far the larger part of the shear. A common rule for the designing of beam webs is to *assume* that the web takes *all* of the shear, and that the unit stresses are *uniformly* distributed over it. In this case, the height of the web is taken as the full depth of the beam.

Applied to the case in hand this rule gives

$$50{,}000 \div (20 \times \tfrac{1}{2}) = 5{,}000 \qquad \text{lbs. per sq. in.}$$

as the *average* shearing unit stress on the beam web. This is about 13 per cent. less than the true value shown in the figure. The assumed stress distribution is indicated by the dotted lines in Fig. 351. The discrepancy between the maximum unit stress given by this

* See note, p. 311.

assumption and the real stress determined from § 140 will vary with the proportions of the beam in question. For the standard I beams, the *true* solution gives results which are from 11 per cent. to 23 per cent. greater than those given by the *assumed* distribution. It is common practice to make allowance for this fact by lowering the allowable unit stress when the rule is used. One rule in common use keeps the allowable stress for steel down to 10,000 lbs. per sq. in. instead of the usual 12,000 lbs. per sq. in.

Such a rule is valuable for quick approximations and for ordinary routine work, but for close designing and with sections not like the standard ones, the method of § 140 should be followed. (See also D below.)

B. Diagonal Buckling of the Web. In § 201, it was shown that the moment and shearing stresses combine to form diagonal stresses of tension and compression, and in Fig. 345, the diagonal *tensile* stresses were shown separately. These stresses were seen

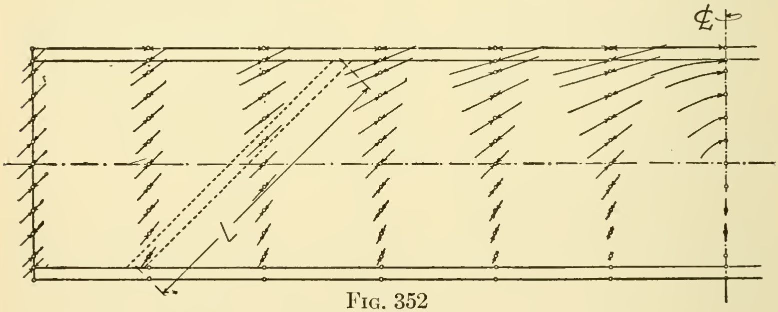


Fig. 352

to be especially dangerous for beams made of materials like wood and concrete that have low tensile strength. The diagonal stresses of *compression* are especially dangerous to beams with thin webs, since they set up column action which tends to buckle the thin webs.

Figure 352 is drawn to show the resultant *compressive* stresses found by combining the moment and shearing stresses in a beam whose section is like Fig. 347 and which carries a concentrated load of 60,000 lbs. at the center of a span of 120″. The equations of § 200 were used in determining the stresses and directions.

In this drawing the arrows show both the direction and amount of the resultant compressive stress at the various points. Near the center the directions change rapidly, due to the rapidly changing shear, and this method of representation becomes inadequate to express the facts. For this reason the arrows at the center section have been curved to indicate the change from an inclined to a horizontal direction. An examination of Fig. 352 will show that throughout the greater part of the web large compressive stresses are present. On any diagonal strip (see dotted lines in Fig. 352), the compressive stresses vary somewhat; but in general they produce unit stresses that may become excessive when the strip is viewed as a column of length L and of a least diameter equal to the web thickness.

However, the strip of web does not act precisely like a column because it is joined to strips which adjoin it and also it is acted upon by the diagonal tension which tends to prevent buckling. Experience shows that failures due to web buckling are easily possible for short, heavily-loaded beams.

No complete rational analysis of this phenomenon has been made, but several formulas for testing the safety of thin webs are in common use.

All of these formulas are based on about the same concept. The maximum shearing stresses, which occur at the neutral surface, combine to produce diagonal compression at 45° to the horizontal. The unit stress due to this diagonal compression at the neutral surface is equal to the shearing unit stress at the same point (§ 200). The diagonal web strip (Fig. 352) is then viewed as a free column under a unit stress *throughout* equal to the shearing unit stress at the neutral surface. Its safety is then tested by some accepted column formula.

This concept involves several rather generous approximations. However, it has proved its usefulness in practice and the results check fairly well with the available test data.

One formula, proposed by C. R. Young, is as follows*: Let the safe *average* shearing unit stress (see A, above) be s_s; let the

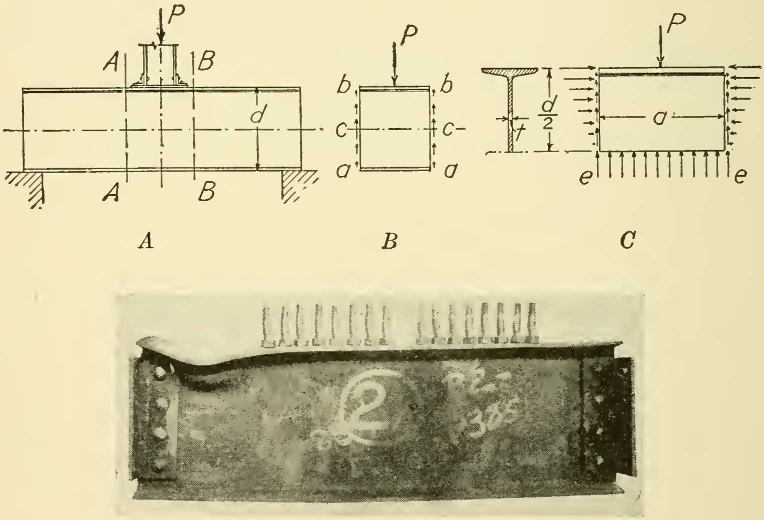
*See Hool and Kinne, *Structural Members and Connections.*

height of the beam web measured between fillets be h and let the thickness of the web be t. Then

$$\text{For ratios of } \frac{h}{t} \text{ between } 0 \text{ and } 60\text{:}\quad s_s = 15{,}000 - 150\,\frac{h}{t},$$

$$\text{For ratios of } \frac{h}{t} \text{ between } 60 \text{ and } 115\text{:}\quad s_s = 10{,}200 - 70\,\frac{h}{t}.$$

C. Vertical Buckling of the Web. When a thin webbed beam carries a heavily concentrated load, that load sets up local stresses which, when combined with the bending and shearing



D

Fig. 353

stresses, may prove the controlling factors in design. As an illustration, take the beam shown in Fig. 353A. Let the part between sections AA and BB be cut free and shown in Fig. 353B. The load P is supported by the shearing stresses on the sections. These stresses vary in intensity from zero, at points a and b, to a maximum at c (§ 143). The top half of the beam is also under compression due to bending, while the bottom half is under tension. Let the top half be cut loose from the bottom and

shown free in Fig. 353*C*. The effect of the shear on the bottom half is represented by the arrows *ee*, and is *assumed* as uniformly distributed. It is evident that the web of this beam is under a very complex set of stresses, and that the tendency to buckle is similar to that in a column. In the lower part of the beam, the tendency to buckle is somewhat neutralized by the tensile stresses due to bending. The same tendency to buckle is present (in the reverse sense) at reactions; and wherever heavily concentrated loads occur. In Fig. 353*C*, the web is then a column of length $d/2$ with a cross section of at sq. in., carrying a load of P lbs.

The formulas used by the American Bridge Company to cover this case are as follows: *

$$\text{Safe interior load} = 2f_b t\left(a_1 + \frac{d}{4}\right),$$

$$\text{Safe end reaction} = f_b t\left(a + \frac{d}{4}\right).$$

In these formulas, t is the thickness of the web, d is the depth of the beam, a is the distance over which an end reaction is distributed, a_1 is one half of the distance over which an interior load is distributed, and f_b is the allowable compressive unit stress as fixed by the web section. This stress is determined from the expression $f_b = 19{,}000 - 173(d/t)$, which is merely a transformation of the American Bridge Company column formula.

Here the factor $d/4$ is apparently an arbitrary one intended to make the formula correspond with the tests.

Figure 353*D* shows a beam which has been tested to failure by a load concentrated near one end.

D. Fillets. An examination of Fig. 352 will show that the unit stress which occurs at the junction of the flange and web may exceed the unit stress on the outermost fiber in the same vertical plane. This will be true only for beams with short spans and heavy loading, in which case large moment stresses are combined with large shearing stresses. It indicates, however, that in such cases these possible maximum stresses must be considered. The

* These formulas are based on unpublished tests.

fillets on the standard I beams tend to reduce these maximum stresses, and they also serve a useful purpose in manufacture.

PROBLEMS

1. (*a*) What is the minimum span on which a 20″ × 65 lb. I beam may be used to carry its full safe load in bending? Use the approximate method of § 204A.
2. Repeat Problem 1, using a 20″ × 100 lb. beam.
3. What is the degree of accuracy of the rule in § 204A, in the case of a 15″ × 42 lb. I beam?
4. What is the greatest uniformly distributed load that may be carried on a 24″ × 100 lb. I section, (*a*) as determined by shearing unit stress, computed accurately, (*b*) as determined by the approximate rule, § 204A, (*c*) as determined by diagonal web buckling?
5. What is the greatest total shear that may be carried on the plate girder section shown in Fig. 241? Use the methods of § 204.
6. What is the greatest safe end reaction on a 15″ × 42 lb. I beam which has bearings which extend 4″ along the length of the beam?
7. How much load may be safely concentrated on a space 6″ long at the center of a 12″ × 31½ lb. I beam?
8. If the section, Fig. 347, is that of a steel beam, what length of end reaction is needed to develop its full strength?
9. If the section, Fig. 347, is to be used for a steel beam, 15′ 0″ long, carrying a concentrated load at the center of the span, what must be the length of the end and center bearings to allow the greatest possible load?

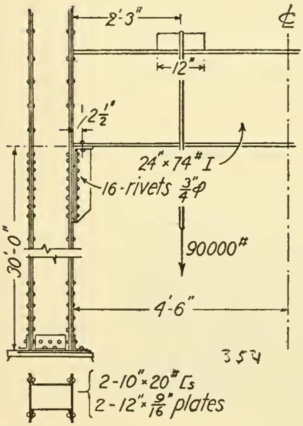
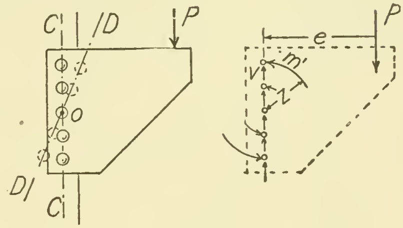


FIG. 354

10. A 20″ × 65 lb. I beam carries a load of 60,000 lbs. at the center of a span of 10′ 0″. The load has a bearing on the beam which extends 6″ along the beam and the bearings at the reactions are 4″ long. Investigate the safety of the beam.
11. Investigate the safety of the construction shown in Fig. 354.

205. Eccentric Riveted Connections. In the fabrication of steel work, it frequently becomes necessary to design connections which are loaded eccentrically. Two principal cases will be noted.

(1) *When the moment due to eccentricity is in the plane of the connection.* This case is illustrated in Figs. 355 and 356. In Fig. 355*A*, the load P tends to rotate the bracket in a clock-wise direction. This moment tends to deform the rivets and to rotate the line of holes CC to some new position DD. If we assume that the deformations follow the law of the maintenance of plane sections (§ 130) which applies to bending moments in beams, the line DD will be a straight line and the deformation of any given rivet (and hence the unit stress on that rivet) will be proportional to its distance from o.



A FIG. 355 *B*

Now take the bracket as a free body, Fig. 355*B*. In order to prevent translation (downward) the rivet group must offer a resistance (upward) equal to P. This can be assumed to be divided equally among the group, as shown by the vertical arrows; then $V = P/n$, where n is the number of rivets in the group.

The moment Pe is resisted by the component stresses on the rivets indicated by m'. If we let M equal the moment stress on a rivet at 1″ from o, the stress on any rivet distant z from o, as noted above, will be $m' = Mz$; and the counterclockwise resisting moment exerted by that rivet will be Mz^2. The entire resisting moment of the group will then be ΣMz^2, and it must equal Pe.

The center of rotation o lies at the center of gravity of the group of rivets.* We, therefore, can investigate the stresses in such a group by computing the moment stress and the direct stress on each rivet and finding the resultant of the two.

Thus in the group shown in Fig. 356*A*, the direct stress on each rivet is $12{,}000 \div 8 = 1{,}500$ lbs. In order to determine the moment

* If proof of this is necessary, let the student consider the principle of least work (§ 238) in connection with the fact that moment of inertia ($\int z^2 dA$) is least about a gravity axis (§ 116 (6)).

stress, the group is laid out at large scale (Fig. 356*B*) and the distances of the rivets 1 and 2 from *o* are scaled. Now if M represents the moment stress on a (hypothetical) rivet at 1″ distance from *o*, then the moment stress on rivet 1 is 6.66M and the moment of

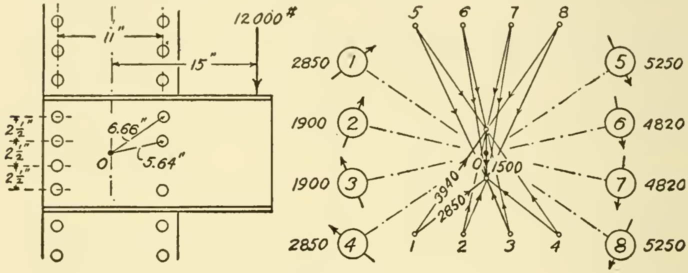


Fig. 356

this stress is $(6.66)^2M = 44.3M$. Similarly for rivet 2, the moment is $31.8M$; and of the whole group (ΣMz^2) is $4(44.3M + 31.8M) = 304.4M$. This must equal the moment of the load; therefore $304.4M = 12{,}000 \times 15$, whence $M = 591$ lbs. ins., which is the moment of the stress on a (hypothetical) rivet, 1″ from *o*. The stress on this rivet would be $591 \div 1 = 591$ lbs. The moment stress on rivet 1 is then $6.66 \times 591 = 3{,}940$ lbs. and on rivet 2 it is $5.64 \times 5.91 = 3{,}335$ lbs., each rounded off to the next larger five pounds.

The moment stress, the direct stress and their resultant on each rivet are shown in the combined stress diagram in Fig. 356*B*, that for rivet 1 being emphasized. If all the rivets in the group are of the same size, the one with the greatest stress controls the safety of the group.

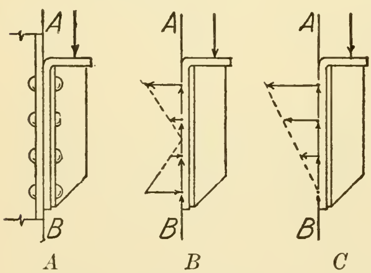


Fig. 357

(2) *When the moment of eccentricity is in a plane perpendicular to that of the connection.* Let Fig. 357*A* show such a case. The principle of a combined direct shear and rotating moment is the same as before, except that in this case the moment sets up

tensile and compressive stresses instead of shearing stresses. If we take the center of the group as the center of rotation, the resisting moment will be built up as shown in Fig. 357*B*, and the stresses can be computed as in the case (1) above. But the bracket bears against the face of the support (along the line *AB*), and so the compressive stresses will be distributed over the surface of contact rather than concentrated in the rivets. This will tend to make the center of rotation move downward (as in Fig. 357*C*). It is sometimes *assumed* that the center of rotation, in extreme cases, will move downward to the bottom rivet in the connection. For any assumed center of rotation, the vertical shearing stresses and the tensile stresses can be computed. In this case, the stresses are in different planes and cannot be added. It is usual to design the joint for either shear or tension and use the larger result.

The use of rivets in tension is to be avoided wherever possible, as the tensile strength of the heads is considered unreliable. Bolts are preferable for such uses.

PROBLEMS

1. The standard beam connection for a 15″ × 42 lb. I beam carries a shear of 20,000 lbs. What is the maximum shearing unit stress on the riveting?
2. What is the greatest shearing unit stress in the connection shown in Fig. 358?

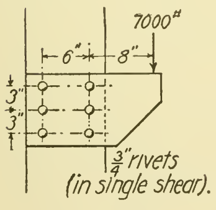


Fig. 358

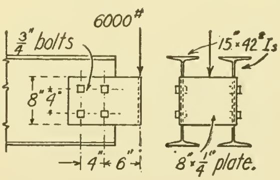


Fig. 359

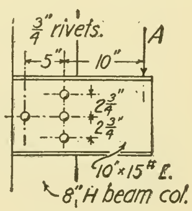


Fig. 360

3. Investigate the connection shown in Fig. 359.
4. What is the safe load *A* on the connection shown in Fig. 360?

CHAPTER XXI

COMBINED MATERIALS

206. Introduction. In structural practice it is frequently desirable to use two or more materials in combination to carry loads. When the different materials are superimposed in the line of stress, as in the column base in Fig. 141, no difficulty is introduced into the design, and the problem may be handled as in Chapters VIII to XI. But when the materials occur side by side in the line of stress, as in a reinforced-concrete column or in a beam, as illustrated in the cases outlined below, a different principle of stress distribution is involved, and it requires special treatment.

In practice such combinations are usually of wood and steel, or concrete and steel, though other combinations are of course possible. The principles developed below may be used for any such cases.

It is well to note that the theory here presented is based on the assumption that the two materials which are used in combination are so thoroughly fastened together that they must deform equally under load. To what extent this condition can be realized or is realized in any given case is often debatable. In the case of reinforced concrete, the natural bond between the materials is quite effective and definite,* but when bolts must be used as in combining wood and steel, the number required and their placing and effectiveness is less certain. The same uncertainty is present, of course, in the results of the computations of stress distribution.

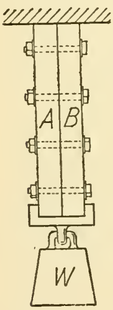


Fig. 361

207. General Principle. Let the bar in Fig. 361 be composed of two materials, as shown, thoroughly fastened together (§ 206)

* Experimental results show that at failure the bond between concrete and steel varies from 200 to 750 pounds per square inch of surface in contact. For good concrete and plain round steel bars 80 lbs. per sq. in. is a fair working stress. (See § 90.)

so that their deformations due to the load W will be equal. Then of course the *unit* deformations are equal, and it follows directly that the unit stresses in the two materials are proportional to their moduli of elasticity.* This is merely another way of saying that the stiffer of two materials offers the greater resistance to deformation.

As an illustration of the application of this principle, let it be required to find the part of the total load carried by each of the parts of the bar in Fig. 362, as well as the unit stress in each. Let x be unit stress in A. Then $20x/12$ is unit stress in B and $18x/12$ is unit stress in C. Then the total stresses are $3x$, $90x/12$ and $27x/12$, respectively. The sum of these must equal the load; hence

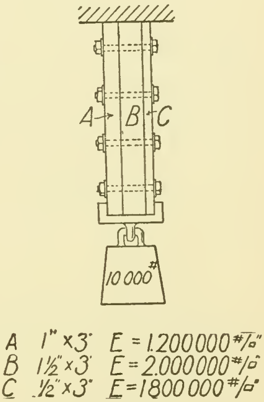


FIG. 362

$$3x + \frac{90}{12}x + \frac{27}{12}x = 10{,}000.$$

Then the unit stress in A is

$$x = 784.3 \qquad \text{lbs. per sq. in.,}$$

the unit stress in B is

$$\frac{20x}{12} = 1{,}307.2 \qquad \text{lbs. per sq. in.,}$$

* If a proof of this statement is desired, let W' represent the part of the load supported by bar A (Fig. 361); let E' be its modulus of elasticity, and A' its cross-sectional area. Also let W'', E'', and A'' represent the same quantities for bar B; and let U represent the unit elongation common to the two. Then by definition

$$E' = \frac{\frac{W'}{A'}}{U}, \qquad U = \frac{\frac{W'}{A'}}{E'}; \qquad \text{and} \qquad E'' = \frac{\frac{W''}{A''}}{U}, \qquad U = \frac{\frac{W''}{A''}}{E''};$$

then

$$\frac{\frac{W'}{A'}}{E'} = \frac{\frac{W''}{A''}}{E''}, \qquad \text{and} \qquad \frac{E'}{E''} = \frac{\frac{W'}{A'}}{\frac{W''}{A''}},$$

which means that the unit stresses in the bars are proportional to their respective moduli of elasticity.

the unit stress in C is

$$\frac{18x}{12} = 1{,}176.5 \qquad \text{lbs. per sq. in.;}$$

and the total stresses are

For A :	784.3×3 =	2,353	lbs.
For B :	$1{,}307.2 \times 4\frac{1}{2}$ =	5,882	lbs.
For C :	$1\ 176.5 \times 1\frac{1}{2}$ =	1,765	lbs.
Total		10,000	lbs.

Evidently the same principle holds good where the stresses are compressive instead of tensile, but in no case can it be applied if the stresses exceed the elastic limit of any of the materials. Nor can it be applied unless the materials are so well bonded, so loaded or so fastened together that they deform equally under stress.

PROBLEMS

1. A bar as in Fig. 362, whose cross section is $1'' \times 2''$ is composed of a central core of steel $\frac{1}{2}'' \times 2''$ and two side plates of brass, each $\frac{1}{4}'' \times 2''$. It carries a tensile load of 20,000 lbs. axially. What is the unit stress in each material?
2. What is the safe load on the bar in Problem 1?
3. A $6'' \times 6''$ timber, $1'\,0''$ long, has two steel plates $\frac{1}{2}'' \times 6''$ bolted to opposite faces of the timber. What is the safe axial load in compression?
4. A block whose cross section is $3'' \times 3''$ is composed of strips of aluminum, brass and bronze, the cross section of each strip being $1'' \times 3''$. It carries a compressive load of 30,000 lbs. axially. What is the unit stress in each strip?
5. What is the safe load in Problem 4?

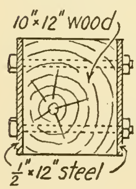


Fig. 363

208. Flitched Beams.* Case I. When two different materials are combined in a beam after the manner shown in Fig. 363, the principle of equal deformations already noticed will afford a means for determining the strength of the beam. Suppose the load, span, and cross section to be given,

*A flitched beam or column is composed of one or more pieces of timber with one or more steel plates or channels bolted on the sides or between the timbers. Formerly such members were much used in bridge building, for derrick booms, etc., but with the lessening cost of steel and the increasing cost of timber, their use is becoming less general. The principle involved in the design of such members is nevertheless of importance.

and that we wish to determine whether the unit stresses set up are within the working values. In this case, the deformations take the form of beam deflections (Chapter XVII) and the part of the total load carried by each material can be determined by working backward from the equation for deflection.

For example, assuming a beam with a cross section like that in Fig. 363, let L denote the span, w the load per foot (uniformly distributed), E_1 and I_1 the moduli of elasticity and the moment of inertia of the steel, and E_2 and I_2 the corresponding quantities for timber. Also let x be the partial load carried by the steel, and y that carried by the timber.

Now the deflection of the steel is equal to deflection of the timber, i.e.,

$$\frac{5}{384}\frac{x}{E_1}\frac{L^3}{I_1} = \frac{5}{384}\frac{yL^3}{E_2 I_2} \qquad \text{or} \qquad \frac{x}{E_1 I_1} = \frac{y}{E_2 I_2},$$

or

(1) $$\frac{x}{y} = \frac{E_1 I_1}{E_2 I_2};$$

moreover

(2) $$x + y = wL.$$

From these two equations, x and y can be determined in terms of known quantities, w, L, E, and I. Now having determined the load carried by each part, the stresses due to that load can be determined in the usual manner.

For instance, let it be required to find the unit stresses set up by a uniformly distributed load of 10,000 lbs. carried on a beam having a span of 20′ 0″ and a cross section like Fig. 363. Using the notation of the preceding example, we have $x + y = 10{,}000$, and

$$\frac{x}{y} = \frac{29{,}000{,}000\left(\dfrac{1 \times 12 \times 12 \times 12}{12}\right)}{1{,}000{,}000\left(\dfrac{10 \times 12 \times 12 \times 12}{12}\right)}.$$

Solving these equations, we find $x = 7{,}436$, $y = 2{,}564$.

Now let the unit stress in the steel be s_1, and in the timber s_2, then

$$\frac{7{,}436 \times 20 \times 12}{8} = s_1 \frac{1 \times 12 \times 12 \times 12}{6 \times 12},$$

and

$$\frac{2{,}564 \times 20 \times 12}{8} = s_2 \frac{10 \times 12 \times 12 \times 12}{6 \times 12},$$

whence

$$s_1 = 9{,}295 \text{ lbs. per sq. in.,} \qquad s_2 = 320.5 \text{ lbs. per sq. in.}$$

Note that these stresses are in proportion to the respective moduli of elasticity (§ 207). Frequently steel channels are used instead of plates, but the principle of the solution remains the same.

Case II. When the timber and steel are combined as in Fig. 364, a different form of solution is necessary. Let us suppose the beam in question to have a span of 15′ 0″ and a load of 8,000 lbs., uniformly distributed, and let it be required to investigate the combination. Since the upper face of the steel plate and the lower face of the timber are assumed to deform equally, the unit stress in the steel at that surface will be twenty-nine times as great as that in the wood. This is equivalent to saying that the steel takes twenty-nine times as much stress as an equal amount of wood similarly placed, and that the actual section shown in Fig. 364 will act the same as a section made entirely of wood and

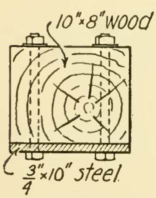


Fig. 364

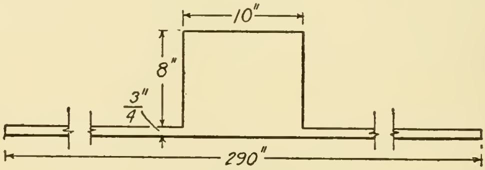


Fig. 365

shaped like Fig. 365. This figure shows what is sometimes called the ***transformed section,*** equivalent to the actual composite section. In such a case, the neutral axis can be located by

finding the center of gravity of this transformed section. The moment of inertia about the center of gravity can be found next and the problem solved as for any beam with an unsymmetric cross section (see § 136), due regard being paid to the fact that in the actual section (Fig. 364) the *unit* stresses in the steel will be twenty-nine times as great as those determined for the flange of the transformed section (Fig. 365).

If the beam in Fig. 364 is used to span 12′ 0″, carrying a distributed load of 8,000 lbs., the unit stress in the steel at its top face will be 2,150 lbs. per sq. in. and at its bottom face will be 4,175 lbs. per sq. in.; while the stresses in the timber will be 668 lbs. per sq. in. at the top and 74 lbs. per sq. in. at the bottom. Let the student check these figures.

NOTE. The preceding cases are selected because they form a good demonstration of the theory of combined materials rather than for their practical aspects. In practice, it is not possible to unite steel and timber so that they will deform absolutely in unison, particularly when they are placed as in Case II above.

PROBLEMS

1. A flitched beam, 14′ 0″ long, is made from an 8″ × 12″ timber with a ½″ × 12″ steel plate bolted to each side. Find the safe uniformly distributed load on the beam and the part of the load carried by each material.
2. Using the same plates and timber as in Problem 1, let the 12″ face of the timber be placed horizontally while the plates remain vertical and centered on the timber. What is the safe uniformly distributed load on the beam?
3. Using the same timber as in Problem 1, let the depth of the plates be changed (width remaining the same) so that each material shall be stressed to its full safe working stress. What is the safe load, uniformly distributed, for the new design?
4. A timber, 8″ × 12″ and 18′ 0″ long, has a 12″ × 20½ lb. channel bolted to each side. How much load, concentrated at the center, may be safely supported by the flitched beam?
5. Using the same beam as in Problem 4, find the maximum unit stress in each material when the beam is loaded with 20,000 lbs., uniformly distributed.
6. Let the timber in Problem 2 be placed so that the 12″ sides are horizontal. How much load may be uniformly distributed over the span with safety?
7. A flitched beam is made of a 10″ × 12″ timber with a 10″ × ½″ steel plate bolted to the top and bottom faces. What is the resisting moment of the section, based on working unit stresses?

8. An $8'' \times 8''$ timber beam has two $\frac{1}{2}'' \times 6''$ steel plates bolted to opposite faces of the timber. Is it stronger when the plates are horizontal or vertical; and by how much?
9. A $10'' \times 12''$ timber, $15'\ 0''$ long, has an $8'' \times \frac{1}{2}''$ steel plate bolted to the top face. What is the safe uniformly distributed load on the beam?

209. Reinforced-Concrete Columns. There are two general types of reinforcement for concrete columns. (See Fig. 366.) In one the steel is in the form of bars placed longitudinally in the column. In the other, the steel, in the form of bands, small bars, or wires, is placed around the outside, horizontally or spirally. When vertical bars are used, it is usual in practice to tie them together transversely in order to keep them properly placed. When transverse or spiral reinforcement is used, longitudinal spacing bars are found useful. Thus it is seldom that either of the typical forms of reinforcement is used alone. The theories covering the two cases are quite distinct, however, and they will be taken up separately, as if each type of reinforcement were used alone.

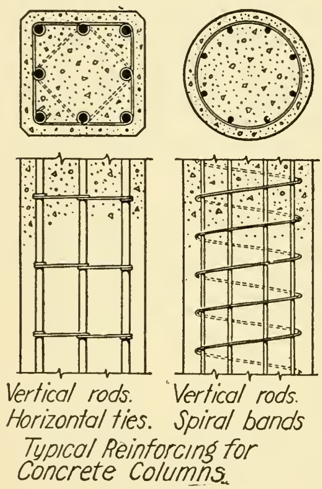


Fig. 366

(*a*) *Longitudinal bars.* A typical cross section of a reinforced-concrete column is shown in Fig. 367. The bars usually are placed near the outside face of the column so as to add as much as possible to the resistance of the column in bending, but enough concrete is placed outside the steel (from $1\frac{1}{2}''$ to $3''$) to protect the steel from corrosion and fire. No satisfactory method is in common use for calculating the strength of a reinforced concrete column as determined by its slenderness ratio. Present practice is to set an arbitrary limit of the ratio of the length to the least

Fig. 367

diameter, corresponding to a short column,* and not to design any columns of a greater slenderness ratio, unless they are justified by experimental results. Data on long concrete columns may be accumulated which may justify the designing of more slender columns than are now commonly used. But for the present, the principles here given can be applied to all columns up to a length of fifteen diameters. Thus the column in Fig. 367 should not be designed for a length over $12\frac{1}{2}$ ft. and for that or any shorter length its strength would be determined as follows.

If the modulus of elasticity of the concrete is 2,000,000 lbs. per sq. in. and that of the steel is 30,000,000, the actual unit stress in the steel will always be fifteen times that in the concrete (§ 207). Then when the concrete is stressed to its working strength (450 lbs. per sq. in.), the steel will be stressed to 6,750 lbs. per sq. in. and the working load on the section in Fig. 367 is

$$\begin{array}{rrl} 4 \times 6{,}750 = & 27{,}000 & \text{lbs.} \\ 96 \times 450 = & \underline{43{,}200} & \text{lbs.} \\ \text{Total} & 70{,}200 & \text{lbs.} \end{array}$$

It should be noted that in this case the steel cannot be stressed to its safe working strength without overstressing the concrete. Such use of a material might be considered uneconomical except for the fact that the steel makes a reinforced column much more tough (as against shock, bending moment, etc.) and much more reliable than a plain concrete column and so justifies the use of much higher working stresses on the concrete than would otherwise be advisable.

The preceding theory is not concerned with the position of the steel in the section and would apply equally well if the steel were placed centrally. Usually the outer $1\frac{1}{2}''$ or $2''$ of concrete is not

* See *Report of the Joint Committee on Standard Specifications for Concrete and Reinforced Concrete.* This committee consists of representatives from the American Society of Civil Engineers, The American Society for Testing Materials, The American Railway Engineering Association, The American Concrete Institute and The Portland Cement Association. The committee has published various progress reports between 1909 and 1921. The latest report can be obtained through any of the societies. The report for 1921 is published in full in Hool and Kinnes' "Reinforced Concrete and Masonry Structures."

figured as contributing to the strength of the column on the theory that it might become disintegrated by fire.

(*b*) *Spiral reinforcement.* Reinforcement which incloses the column within horizontal or spiral bands functions quite differently. The concrete under compression tends to expand laterally (to burst). (See § 78.) This lateral motion is prevented by the steel bands which are thus put in tension. Failure of such a column occurs when the bands, becoming overstressed, undergo so much elongation that the concrete escapes laterally. Manifestly, this type of reinforcement is most effective for columns of a circular cross section.

The design of such columns is based almost wholly on experimental results and the consequent rules of practice which are given in texts on reinforced concrete design. A very useful form of column which depends on this principle is made by filling a light iron or steel pipe with concrete.

PROBLEMS

1. A reinforced concrete column is 16″ square and contains reinforcing rods whose aggregate cross section is 5 sq. in. What is the safe load provided $1\frac{1}{2}''$ is allowed all around for fireproofing?
2. A piece of steel pipe, 6″ in diameter outside and $5\frac{1}{2}''$ inside, is filled with concrete and used as a column. If this column carries a concentric load of 30,000 lbs., what is the compressive unit stress in each material?
3. What is the safe load on the column in Problem 2?
4. A reinforced concrete column is $14'' \times 14''$ with four steel rods each $\frac{3}{4}''$ in diameter. It carries a load of 95,000 lbs. What is the unit stress in each material?
5. Design a square reinforced concrete column to carry a load of 96,000 lbs. Let the cross section of the column be made up of 2 per cent. steel and 98 per cent. concrete. Allow $1\frac{1}{2}''$ all around for fireproofing.
6. Design a reinforced concrete column with a circular cross section, 2 per cent. of the cross section being steel. The load is 62 tons. Allow $1\frac{1}{2}''$ for fireproofing.

210. Reinforced-Concrete Beams. The study of reinforced-concrete beams is a very complex one, because of the large number of variables entering into the problem. The literature of the subject has developed a well-recognized set of symbols for these

quantities, given below.* These symbols will be used for this discussion in place of those in the appendix.

The controlling factors in the design of reinforced-concrete beams are the physical properties of the two materials. For the purposes of this discussion, we will use the values given below.† A glance at these values shows that an unreinforced-concrete beam would be very weak on the tension side. Since steel is relatively expensive and very strong in tension, it is ordinarily used only on the tension side of the beam, as in Fig. 368.‡ Such a beam carrying a very light load (so light that on the tensile side the concrete is stressed below its ultimate strength in tension) would act like the flitched beam in Fig. 364. Its *transformed section* (page 328) would look like Fig. 369, the neutral surface being below the center of the depth, as shown, and the distribution of stress on the cross section would be as in

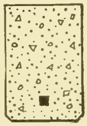


Fig. 368

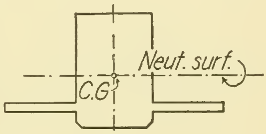


Fig. 369

* A .... Area of steel.
b .... Breadth of beam.
C .... Total compressive stress on concrete.
d .... Depth of beam, to center of steel.
e_s .... Elongation of steel due to f_s.
e_c .... Shortening of concrete due to f_c.
E_s .... Modulus of elasticity, steel.
E_c .... Modulus of elasticity, concrete.
f_s .... Unit stress in steel.
f_c .... Unit stress in concrete.
j .... Ratio of lever arm of stress couple to d.
k .... Ratio of depth of neutral axis to d.
M .... Bending or resisting moment, in general.
M_c .... Resisting moment, determined by concrete.
M_s .... Resisting moment, determined by steel.
n .... Modulus of elasticity ratio, E_s/E_c.
p .... Steel ratio, $A/(bd)$.
T .... Total tension in steel.
V .... Total vertical shear.
v .... Unit vertical or horizontal shear.

† Modulus of elasticity, concrete: 2,000,000 lbs. per sq. in.
Modulus of elasticity, steel: 30,000,000 lbs. per sq. in.

	Tension	Compression
Ultimate strength, concrete:	200	2,000
Working strength, concrete:	20	600
Working strength, steel:	15,000	15,000

‡ In some special cases, it is deisrable to place reinforcement both at the top and bottom of a beam.

Fig. 370. As the load on the beam is increased, the unit stresses increase; and very soon the ultimate strength of the concrete in tension is reached (about 200 lbs. per sq. in.). Now (since $E_s/E_c = 15$) the unit stress in the steel is $15 \times 200 = 3{,}000$ lbs.

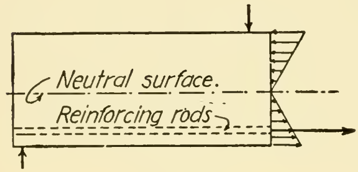


FIG. 370

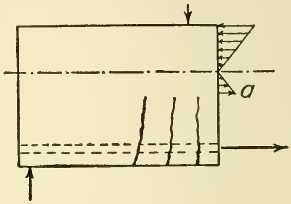


FIG. 371

per sq. in., which is very low for the steel.* If the loading is continued, the concrete must fail in tension, causing cracks which extend upward from the lower face of the beam, as shown in Fig. 346. During this time, the neutral surface must be moving upward, and the new stress distribution diagram is as in Fig. 371.

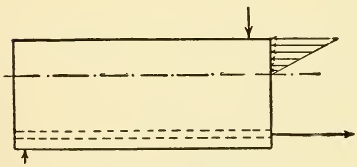


FIG. 372

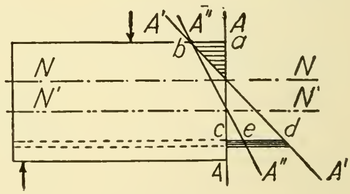


FIG. 373

This is the actual working condition in most reinforced-concrete beams. But in order to simplify the calculations, it is usual to neglect the tensile stresses in the concrete (Fig. 371). This is justified on the grounds that (1) they are small in actual amount and have a small lever-arm about the neutral surface, so that they produce but a very small resisting moment; and (2) if neglected in the calculations but actually present in the beam, they constitute in effect an added factor of safety rather than an element of danger. The assumed stress distribution will then be that shown in Fig. 372.

* This statement assumes that E_c is constant up to the ultimate strength, which is not true. However, no calculations are based on the assumption. The principle remains true even though the exact figures may be in error.

It will be obvious from a glance at Figs. 370–372 and from the preceding paragraph that any determinations of the strength of such a beam must depend upon our knowing where the neutral surface lies in any given case. In fact, this is the key to the entire problem. The fact of the maintenance of plane sections during bending holds good for reinforced-concrete beams as well as for homogeneous beams; and in general the same limitations apply to this whole discussion as to the previous one (§ 132).

In Fig. 373, let AA represent a plane section before bending, and let $A'A'$ represent the same section after the load has been applied. There is then a linear compression of ab in the extreme fiber of the concrete and an elongation of cd in the steel. The neutral surface is at NN. Now let us suppose that the *amount* of steel in the beam is increased, all other factors remaining the same. Obviously the deformation cd will decrease to some amount, ce; the plane section will take the position $A''A''$, and the neutral surface will be at $N'N'$. Again going back to the conditions which give the section $A'A'$, let the steel merely become *stiffer*, i.e., with a greater modulus of elasticity (all other factors remaining the same); again the deformation cd will decrease and the neutral surface NN will fall. Similarly one could show that starting with any assumed conditions, if the *relative* amount of steel to concrete or the *relative* stiffness of steel to concrete is changed, then the neutral surface is changed. Relatively *more* steel or *stiffer* steel will *lower* the neutral surface, and vice versa. It appears then that the location of the neutral surface will eventually depend on the *relative* quantities and qualities of the materials employed; and that k can be expressed in terms of p and n.

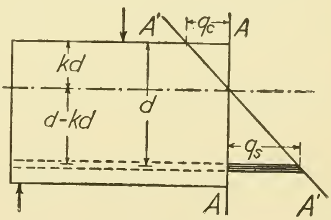


Fig. 374

Let Fig. 374 represent a section cut from a beam under stress. The loads, reactions, and shearing stresses are omitted because they do not affect the present problem. Let AA represent a

plane section, before bending, and let $A'A'$ represent the same section after bending. From similar triangles, we have

(3) $$\frac{q_s}{q_c} = \frac{d - kd}{kd}.$$

But the deformations q_s and q_c are proportional to the unit stresses which produce them, and inversely proportional to the moduli of elasticity of the materials. Therefore we may write

(4) $$\frac{q_s}{q_c} = \frac{E_c}{E_s}\frac{f_s}{f_c} = \frac{1}{n}\frac{f_s}{f_c}.$$

Substituting this value in equation (3), we find

(5) $$\frac{1}{n}\frac{f_s}{f_c} = \frac{d - kd}{kd}.$$

Solving this equation first for f_c, then for f_s, we have

(6) $$f_c = \frac{f_s k}{n(1 - k)},$$

(7) $$f_s = \frac{nf_c(1 - k)}{k}.$$

Turning to Fig. 375, let C represent the total resultant compression on the section, and let T represent the resultant tension. Then, since ΣH must be 0,

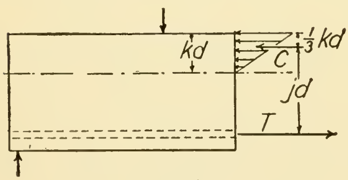


FIG. 375

(8) $$T = C,$$

and we also know that

(9) $$T = Af_s,$$

(10) $$C = \tfrac{1}{2}bkdf_c.$$

Substituting these values in equation (8), we find

(11) $$Af_s = \tfrac{1}{2}bkdf_c.$$

Substituting the value for f_s from equation (7) in equation (11), we get

$$\frac{Anf_c(1-k)}{k} = \tfrac{1}{2}bkdf_c,$$

$$2\frac{An(1-k)}{bd} = k^2.$$

But, by definition,

$$\frac{A}{bd} = p;$$

hence

$$k^2 = 2pn(1-k).$$

Solving this for k, we have

$$k = \sqrt{2pn + \overline{pn}^2} - pn, \tag{12}$$

which gives the location of the neutral surface when the relative quantity p and the relative quality n of the materials are known. Curves (2) in Fig. 376 are drawn to exhibit the relations of k, p, and n, through the values commonly found in practice.

If we take a center of moments on the line of T, Fig. 375, the resisting moment of the beam is seen to be Cjd, if the center is taken on C, the resisting moment is Tjd. In these expressions, jd is the lever arm of the stress couple (compare § 203), and its value is seen to depend on that of k as in the following equations:

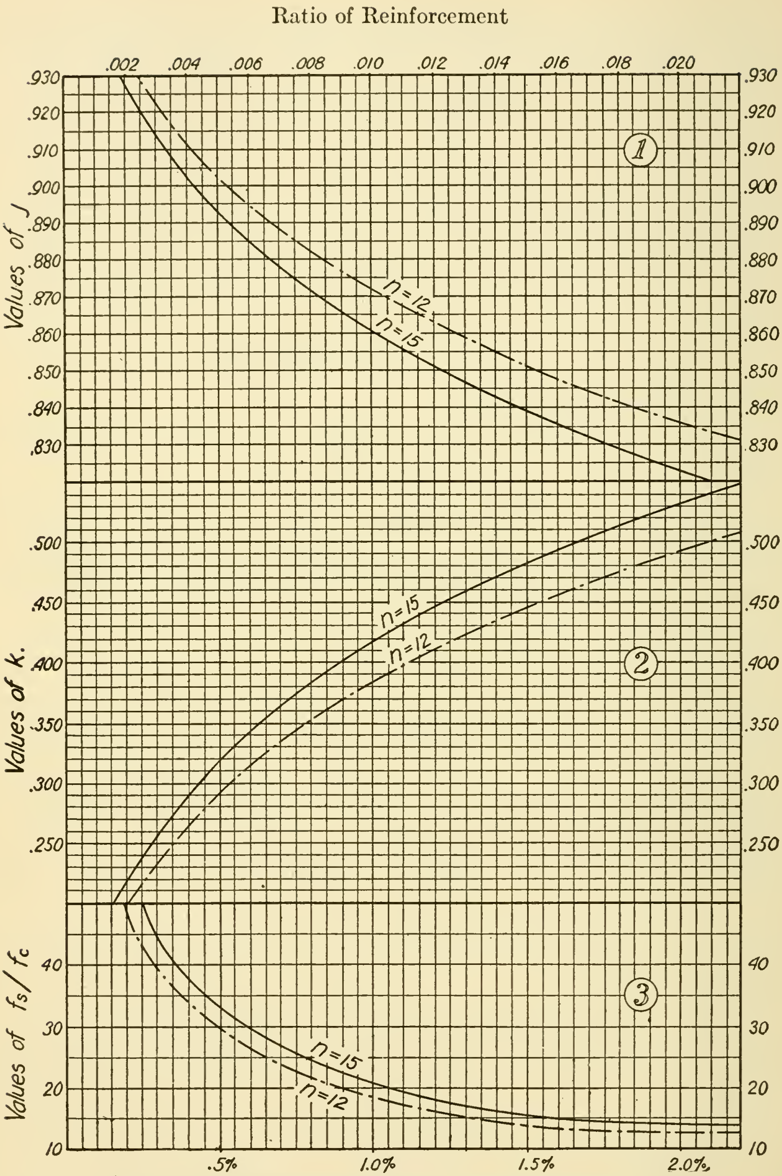
$$\left\{\begin{aligned} jd &= d - \tfrac{1}{3}kd, \\ j &= 1 - \tfrac{1}{3}k. \end{aligned}\right. \tag{13}$$

The curves 1, Fig. 376, display values of j through the range of ordinary practice.

In any beam the bending moment of the loads is equaled by the resisting moment in the beam. In this case this gives rise to the relation

$$M = Cjd = Tjd,$$

in which M can stand for either the bending moment or the resisting moment. Substituting the values of C and T from equations (9) and (10), we find



Curves of the principal equations for Rectangular Reinforced Concrete Beams

Fig. 376

(14) $$M = \tfrac{1}{2}jkbd^2f_c,$$

(15) $$M = Af_sjd.$$

These equations give the relation which exists between an external bending moment and the unit stress in the concrete or in the steel. They may be used precisely like equation (1), § 135.

211. Investigation. An Example. Let the beam in Fig. 377 be placed on a span of 16′ 0″ and carry a uniformly distributed load of 10,000 lbs. In order to determine the unit stresses in the concrete and in the steel we may write (see note, p. 333)

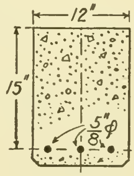


FIG. 377

$$n = 15,$$
$$A = 3 \times 0.3068 = 0.9204,$$
$$p = \frac{0.9204}{12 \times 15} = 0.00511.$$

Using these values in equation (12) (or curve *2*, Fig. 376), we find

$$k = 0.3223,$$

and from equation (13) (or curve *1*, Fig. 376), we have

$$j = 0.8925.$$

Then, from equation (14), we find

$$\frac{10{,}000 \times 16 \times 12}{8} = \tfrac{1}{2} \times 0.892 \times 0.322 \times 12 \times 15^2 f_c,$$

$$f_c = 619 \qquad \text{lbs. per sq. in.;}$$

and similarly, from equation (15):

$$f_s = 19{,}500 \qquad \text{lbs. per sq. in.}$$

From the above, and by reference to the diagrams on p. 338, it is seen that once the theory governing the location of the neutral surface is mastered, the investigation of such a beam becomes a very simple matter.

If we examine the preceding results, however, we see that the steel is badly overstressed, while the concrete is working close to

the allowable stress. This evidently means that the beam needs more steel to make it work most advantageously: it is *under-reinforced.* Now let us substitute three 1″ round bars for the $\frac{5}{8}''$ round bars in the above problem. Then

$$n = 15, \qquad A = 3 \times 0.7854 = 2.3562,$$
$$p = \frac{2.3562}{12 \times 15} = 0.0131,$$
$$k = 0.46, \qquad j = 0.847,$$
$$f_c = 457 \qquad \text{lbs. per sq. in.},$$
$$f_s = 8{,}032 \qquad \text{lbs. per sq. in.}$$

In this case neither material is stressed to its safe strength, but the concrete is working at about three-fourths of its safe stress, while the steel is stressed to but little over one-half its safe stress. That is, the beam is *over-reinforced.*

212. Safe Load. Let it be required to find the full safe load for the second beam in the previous paragraph. The values of n, A, p, k, and j are as before. Now equation (14), § 210, will stand for the resisting moment if we substitute the working strength of concrete for f_c. This gives

$$M = \tfrac{1}{2} \times 0.847 \times 0.46 \times 12 \times 15 \times 15 \times 600$$
$$= 315{,}430 \qquad \text{lbs. ins.},$$

which is the resisting moment of the beam, *as determined by* the strength of the concrete. Again, from equation (15), we have

$$M = 2.356 \times 15{,}000 \times 0.893 \times 15 = 473{,}375 \qquad \text{lbs. ins.},$$

which is the resisting moment *as determined by* the strength of the steel.

Evidently the resisting moment of the beam is the lesser of these values and the safe uniformly-distributed load for the given span is the value of W given by the equation

$$315{,}430 = \frac{W \times 16 \times 12}{8},$$

whence

$$W = 13{,}143 \text{ lbs.}$$

PROBLEMS

NOTE: In the following problems the depth of the beam is to be construed to mean the depth to the center of the steel. See d, Fig. 374.

1. A reinforced-concrete beam is 20″ wide and 40″ deep (to the center of the steel). Its reinforcement has an aggregate cross-sectional area of $7\frac{1}{2}$ sq. in. The span is 16′ 0″ and the load is 8,000 lbs. per ft. What are the unit stresses in steel and concrete?
2. A reinforced concrete beam is 12″ × 24″ and has $1\frac{3}{4}$ sq. in. of reinforcement. The span is 18′ 0″ and the load is 25,000 lbs., uniformly distributed. What are the unit stresses in steel and concrete?
3. A reinforced concrete slab spans 8′ 0″. It is reinforced with 0.42 sq. in. of steel per foot of width and the depth of the slab is 4″. The live load is 100 lbs. per sq. ft. of slab. What are the unit stresses in the steel and in the concrete?
4. What is the safe load, concentrated at the center, on a reinforced concrete beam whose span is 16′ 0″ and whose cross section is 12″ wide and 20″ deep with $2\frac{1}{2}$ sq. in. of steel reinforcement?
5. If the beam in Problem 4 has 1.35 sq. in. of reinforcement, what is the safe load?
6. If the beam in Problem 4 has 1.8 sq. in. of reinforcement, what is the safe load?
7. If the beam in Problem 1 is made of a concrete whose modulus of elasticity is 2,500,000 and a steel whose modulus of elasticity is 30,000,000, what will be the unit stress in each material?
8. If the beam in Problem 4 is made of concrete and steel having a ratio of modulus of elasticity of 13 and safe working strengths of 450 and 13,500 lbs. per sq. in., what is the safe load?

213. Critical Steel Ratio. In §§ 211 and 212, we have considered two beams, one under-reinforced, the other over-reinforced. Evidently, by choosing the proper amount of reinforcement (somewhere between the two amounts already tested) it is possible to produce a beam whose resisting moment is the same whether determined by the strength of the concrete or by the strength of the steel. Moreover such a beam has, in general, an economical design, since in that case each material is working to its full capacity. In order to fix the percentage of steel which will give such a result, we can take equations (14) and (15) (§ 210), put one equal to the other, and solve for $A/(bd)$. This gives

$$\frac{1}{2}jkbd^2f_c = Af_sjd, \qquad (16)$$

(17)
$$\frac{A}{bd} = \frac{k}{2f_s/f_c}.$$

Solving equation (5) (§ 210) for k, we get

$$k = \frac{n}{n + \dfrac{f_s}{f_c}}.$$

Substituting this value of k in equation (17), we find

(18)
$$\frac{A}{bd} = p = \frac{n}{2\dfrac{f_s}{f_c}\left(n + \dfrac{f_s}{f_c}\right)}.$$

This expression gives the relative quantities of the materials (p) in terms of their relative qualities (n and f_s/f_c). When the qualities are known and the quantities are adjusted according to (18), the two resisting moments will be equal, by (16). This is the economical condition we set out to establish.

Curve *3*, Fig. 376, exhibits the values satisfying this equation through the usual range.

214. Design. Logically the first step in the design of a reinforced-concrete beam is to select the materials to be used, i.e., to determine the qualities so that we may start with known values for the working stresses and moduli of elasticity. For the materials in most common use, the ratio n of moduli of elasticity is about fifteen, and the ratio of working stresses (f_s/f_c) is about twenty-five. This establishes our point of attack on the problem.

From equation (18) (or curve *3*), we can establish our critical steel ratio as $0.0075 = p$. We will then make that ratio a condition of the design. It follows, by equations (12) and (13) (or by curves *2* and *1*), that $k = \frac{3}{8}$ and $j = \frac{7}{8}$. We have thus established the location of the neutral surface and the lever-arm of the stress couple for any series of problems, *provided* we use materials corresponding to the above ratios (15 and 25) and *provided further* that we make it a condition of the design that we will use $\frac{3}{4}$ per cent. of steel reinforcement in the beam.

Now let it be required to design a beam (using materials as above) to carry a distributed load of 10,000 lbs. on a span of 16′ 0″. The bending moment is

$$\frac{10{,}000 \times 16 \times 12}{8} = 240{,}000 \quad \text{lbs. ins.}$$

Then, from equation (14) (§ 210), we have

$$240{,}000 = \tfrac{1}{2} \times \tfrac{7}{8} \times \tfrac{3}{8} \times bd^2 \times 600,$$

whence $bd^2 = 2{,}438$. Any values of b and d which satisfy the above equation will serve our purpose.* If we choose 15″ for d, then $b = 10''$ (about). Now the area of the steel must be 0.0075 of that of the concrete. Therefore

$$\frac{A}{15 \times 10} = 0.0075, \qquad A = 1.125 \text{ sq. in.}$$

This quantity of steel can be supplied by using three bars, each $\frac{3}{4}''$ diameter. Moreover these bars can be arranged in a beam 10″ wide to give proper clearances. Two inches of additional width may well be included as fire protection.† Our design then stands as in Fig. 378.

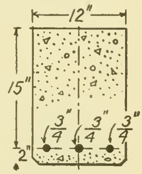


Fig. 378

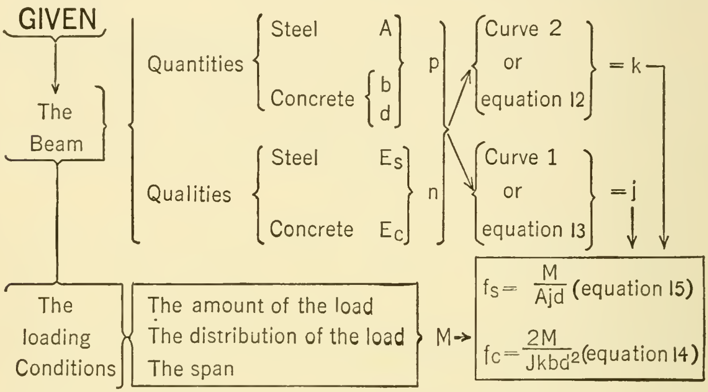
In the above design, equation (15) could have been used in place of equation (14), since either one gives the resisting moment of the beam. Let the student carry the same design through using equation (15).

215. Summary.—Bending on Rectangular Beams. The various equations developed in §§ 210–214 give the means for investigating or designing rectangular beams of reinforced concrete, in so far as the bending stresses are concerned. Because of the large number of variables involved, it requires a number of equations to state the relations which govern the determination of stresses or the design of the beam. It is sometimes difficult at

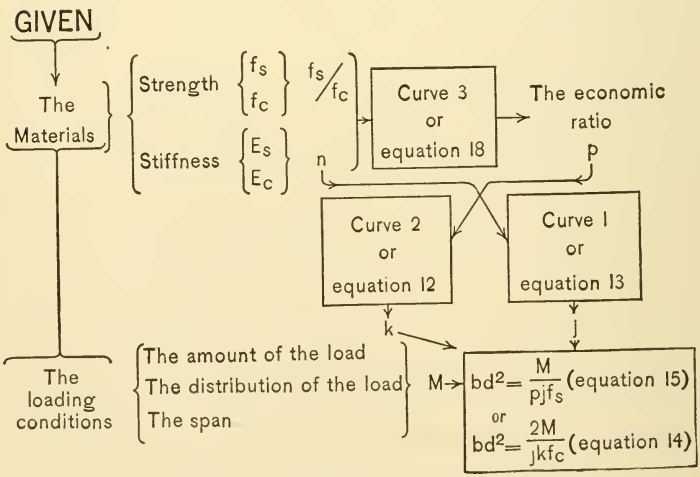
* Compare with design of wooden beams, § 136A.

† The matters of clearances, fire protection, etc., cannot be treated here. Practice has set definite rules governing these points, which can be found in any book on reinforced-concrete design.

first to see the process as a whole because of the large number of parts. Therefore the following diagrams have been prepared, not with the thought of presenting an easy way of making blind substitution, but as a visual aid to comprehending the operation as a whole, showing all the variables, their significance, and their interrelations as they enter into investigation or design.



To Investigate a Reinforced Concrete Beam.



To Design a Reinforced Concrete Beam.

PROBLEMS

1. Design a reinforced-concrete beam to carry a total load of 12,000 lbs. on a span of 15′ 0″. Let the breadth of the beam be 12″.
2. A reinforced-concrete slab spans 12′ 0″. It carries a load of 600 lbs. per sq. ft. besides its own weight. Determine the depth and amount of reinforcement required.
3. A reinforced-concrete beam is to span 19′ 0″ and carry a load of 1,000 lbs. per ft. in addition to its own weight. It is to be 12″ wide. Determine the depth and amount of reinforcement.
4. A certain concrete has a modulus of elasticity of 1,800,000 and safe compressive strength of 400 lbs. per sq. in. A certain steel has a modulus of elasticity of 25,000,000 and a safe tensile strength of 13,000 lbs. per sq. in. Design a beam to be made of these materials to span 18′ 0″ and carry a concentrated load at the center of 8,000 lbs.
5. Using the materials in Problem 4, design a slab to span 5′ 0″ and carry a live load of 150 lbs. per sq. ft.

216. Shear. The principles used in § 140 to develop the shearing unit stress in a homogeneous beam hold equally good for reinforced concrete. In Fig. 379, a section cut from a rein-

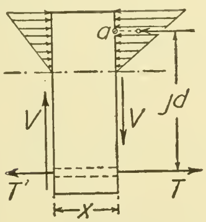


FIG. 379

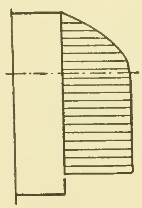


FIG. 380

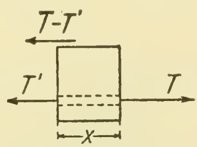


FIG. 381

forced-concrete beam under stress is shown as a free body. (Compare Fig. 246.) From this diagram it will be evident that the variation of the horizontal (or vertical) unit shearing stress will be as shown in Fig. 380 (see § 143).

From Fig. 379, it will be evident that the unbalanced tension $(T - T')$ will cause a slipping of the reinforcing bar unless the bond between the bar and the concrete is strong. Again if we take any part of Fig. 379 below the neutral surface as a free body, as in Fig. 381, it will be seen that the horizontal shear on its top face is equal to $T - T'$ and that the intensity of the shearing stress is

$$v = \frac{T - T'}{bx}. \tag{19}$$

Again in Fig. 379, taking a center of moments at a, we find

$$Vx = (T - T')jd. \tag{20}$$

Substituting the value of $T - T'$ from (19) in (20), we have

$$v = \frac{V}{bjd}.$$

This expression gives the value of the unit shearing stress in beams reinforced with horizontal rods only. Because of the diagonal tension (see §§ 200–201), it is usual to limit shearing stresses determined as above to values between 0.02 and 0.03 of the ultimate compressive strength of the concrete.

When a satisfactory beam cannot be designed for shear as above, it is usual to introduce vertical or inclined steel especially to take care of the diagonal tensile stresses. For a full treatment of shear reinforcement, the student should refer to some text which deals solely with reinforced-concrete design.

PROBLEMS

1. What is the maximum shearing unit stress on the concrete of the beam in Problem 1, § 212?
2. If the beam in Problem 1, § 212, is not specially reinforced to take care of the shear, what is the safe load on the beam as determined by shear?
3. In the beam in Problem 1, § 212, what is the maximum unit stress tending to break the bond between the concrete and the steel reinforcement?

CHAPTER XXII

UNSYMMETRIC BENDING

217. Introduction. The ordinary theory of bending, which was discussed in Chapters XIV to XVIII and XX, applies only to cases of symmetric bending, as defined in § 131.

When the forces producing bending lie in a plane which is not a plane of symmetry of the beam itself, the case is said to be one of unsymmetric bending, and the ordinary theory does not apply. This will become more clear by reference to Fig. 382*A*, which represents a Z bar bent by terminal couples which lie in a vertical plane. Let the part of the bar to the right of *A–A* be taken as

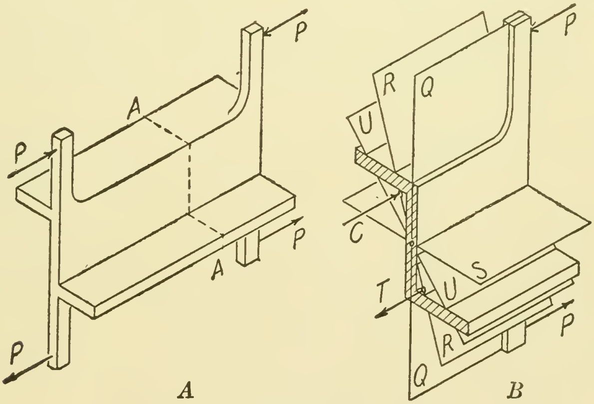


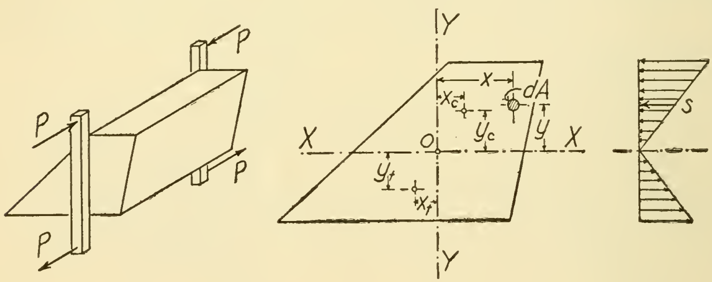
Fig. 382

a free body, as shown on an enlarged scale in Fig. 382*B*. Here the plane Q is the plane of the bending moment. If this case be treated by the *ordinary* theory of bending, the neutral surface is the horizontal plane S and the resultant stresses in tension and compression are as shown by the arrows T and C. Then the plane of the *resisting* moment is the plane R. But obviously if there is to be equilibrium, the bending moment and the resisting moment must lie in the same plane. As a matter of fact in such

a case, equilibrium is maintained by a shifting of the neutral surface to some such position as U; the resultant compression C moving to the right and the resultant tension T to the left until they lie in Q.

In such a case, it is evident that the moment of inertia of the section referred to the neutral surface U is quite different from that referred to S, and also that the distance of the outermost fiber is different. In other words the section modulus I/c for the given beam is very different when referred to the different planes under discussion. In fact, in every case of unsymmetric bending, the real value of I/c depends not only on the size and shape of the section but also on the position of the plane of the external moment.

218. Components of the Resisting Moment. In order to study this question more closely, let us consider the bar shown in Fig. 383A. The cross section of the bar is shown (enlarged) in Fig. 383B. Here the axis YY is the trace of the plane of the external



A *B* *C*

FIG. 383

moment on the cross section. Let us now *assume,* for the purpose of discussion, that XX, which passes through the center of gravity of the cross section o, and which is perpendicular to YY, is the neutral surface of the bar.

The distribution of the unit stresses on the cross section is as shown in Fig. 383C. Let us further assume that the resultant tension T and compression C act at the points t and c, which are described by coordinates as shown. Then the resultant external

moment M produced by the forces in Fig. 383A, is

$$M = Ty_t + Cy_c.$$

But the forces T and C *also produce a moment* about YY which must be zero if the resisting moment is to lie in the same plane as the bending moment, i.e., in the plane YY. It is necessary therefore to find the conditions which will make this resultant moment assume the zero value.

Let s be the unit stress at *any* point (x, y), and let s_1 be the unit stress on an elementary area distant one unit from the neutral surface, which is (assumed to be) in XX. Then $s = s_1y$, the *total* stress on the elementary area is $s_1y\,dA$, and the moment of this stress about the axis YY is $s_1xy\,dA$. Hence the total moment of all the stress on the section, about the axis YY is $\int_A s_1xy\,dA$. This expression is zero when $\int_A xy\,dA$ is zero.

We can therefore say that if XX is the neutral surface of the beam (Fig. 383B), $\int_A xy\,dA$ for the section must be zero. This is an illustration of the general principle that *the neutral surface of a section will be perpendicular to the plane of the external moment if, and only if,* $\int_A xy\,dA$ *for the section* (one axis of reference being in the plane of the moment) *is zero.*

This quantity, $\int_A xy\,dA$ is called the ***product of inertia.**** It will be shown later that the product of inertia is zero for any section which is symmetric about at least one axis (§ 220A). It then becomes evident that the ordinary theory of bending applies to such sections and that for the other cases some means must be found to locate the neutral surface. It will be found that in general the neutral surface, for unsymmetric sections, is inclined at an angle other than 90° from the plane of the external moment. But in order to determine this angle, we must first establish certain principles regarding the moment of inertia and the product of inertia. We shall develop these principles in §§ 219–228. The general question of unsymmetric bending is resumed in § 229 at the point where we leave it here.

* This quantity has no physical significance and the name is chosen in quite an arbitrary fashion (p. 147).

219. Product of Inertia. The product of inertia of an area has been defined in § 218 as $\int_A xy\,dA$ for the given area referred to a pair of fixed axes. In order to get the significance of this expression, let us consider the area and the axes shown in Fig. 384. Let the elementary strip of area dA (shown shaded) be chosen and let x and y be the coordinates of the center of gravity of the strip. Now it is evident that the static moment (§ 53) of this strip, about the Y axis, is $x\,dA$ and that $xy\,dA$ will then represent

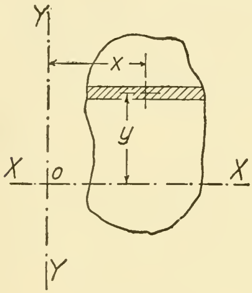


Fig. 384

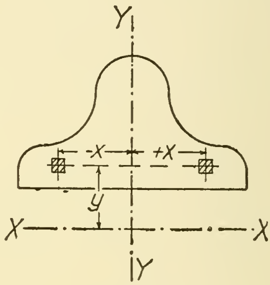


Fig. 385

this static moment times y. Therefore, the product of inertia of an area $\int_A xy\,dA$ can be regarded as the static moment of the whole area, figured about *either* axis, times the distance of its center of gravity from *the other* axis; since this argument can be restated with x and y interchanged. This approach to the problem will be used in the following solutions. Hereafter the symbol K will be used to denote product of inertia.

220. Product of Inertia.—Simple Geometric Figures. A. Any Figure Symmetric about one of the Axes. Let Fig. 385 represent any figure which is symmetric about *one* of the axes of coordinates (in this case YY). It is evident that for any element whose K is $(+x)y\,dA$, there is another whose K is $(-x)y\,dA$. The same would have been true if the figure had had its axis of symmetry on the XX axis of coordinates. Therefore the K for the entire figure is zero.

B. Rectangle.—Origin at Corner. Using Fig. 386, let dA be the strip shown shaded. Then $dA = b\,dy$, the static moment of dA, about YY, is $(b\,dy)(b/2) = (b^2/2)dy$, and

$$K = \int_0^d \frac{b^2}{2} y\,dy = \frac{b^2}{2}\int_0^d y\,dy = \frac{b^2d^2}{4} = \text{area} \times \frac{bd}{4}\cdot$$

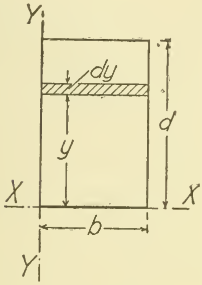


Fig. 386

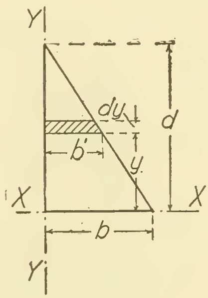


Fig. 387

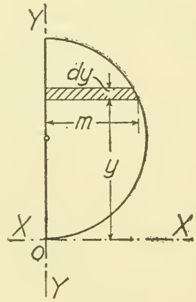


Fig. 388

C. Right Triangle. Using Fig. 387, let us choose the elementary strip as shown. From similar triangles, $b' = b(d - y)/d$. Then the static moment of the strip about YY is

$$\frac{b^2}{2d^2}(d - y)^2 dy$$

and

$$K = \int_0^d \frac{b^2}{2d^2}(d - y)^2 y\,dy = \frac{b^2d^2}{24} = \text{area} \times \frac{bd}{12}\cdot$$

In cases B and C, it may be interesting to note that the form of the answer shows that either dimension of the figure may be chosen as the base.

D. A Semi-circle. Referring to Fig. 388, we see that the elementary area is $dA = m\,dy$. The static moment of this area about YY is $m^2\,dy/2$ and

$$K = \int_A \frac{m^2}{2} y\,dy.$$

But from the equation of a circle referred to o, we have $m^2 = 2ry - y^2$. Then

$$K = \int_0^{2r} \frac{2ry - y^2}{2} y\,dy = \left(\frac{ry^3}{3} - \frac{y^4}{8}\right)\Big]_0^{2r} = \frac{2r^4}{3}\cdot$$

PROBLEMS

1. Find, by integration, the product of inertia of the rectangle in Fig. 386, referred to the upper and left-hand edges.
2. Find, by integration, the product of inertia of the triangle in Fig. 387, referred to H and V axes which intersect at the apex of the triangle.
3. Find, by integration, the product of inertia of a circle which is tangent to both of the axes of coordinates.

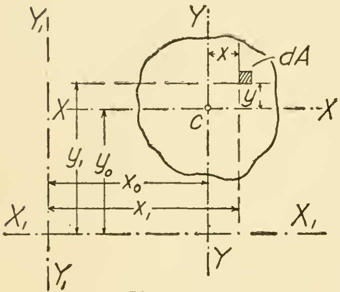


FIG. 389

221. Transfer between Parallel Axes. Let Fig. 389 represent an area whose center of gravity is at c and whose product of inertia, with respect to the axes through c, is K. Let it be required to find the product of inertia K_1 with respect to the axes X_1X_1 and Y_1Y_1. Then we have

$$K = \int_A xy\,dA, \qquad K_1 = \int_A x_1y_1\,dA.$$

But $x_1 = x + x_0$ and $y_1 = y + y_0$; hence

$$K_1 = \int_A (x + x_0)(y + y_0)dA$$

$$= \int_A (xy + xy_0 + x_0y + x_0y_0)dA.$$

But x_0 and y_0 are constants; therefore

$$K_1 = \int_A xy\,dA + y_0\int_A x\,dA + x_0\int_A y\,dA + x_0y_0\int_A dA.$$

Since c is the center of gravity of the area, $\int_A x\,dA$ and $\int_A y\,dA$ are zero (§ 52). Therefore

$$K_1 = \int_A xy\,dA + x_0y_0\int_A dA,$$

(1) $$K_1 = K + x_0y_0A.$$

Compare this result with that of § 113.

In the preceding equation, the values of x_0 and y_0 may be interchanged without affecting the results. This evidently means that, if the value of K for a given figure is transferred from one pair of axes to a second pair, there will be a third pair of axes about which K will be the same as for the second; and that the second and third pair are symmetrically placed with regard to the first.

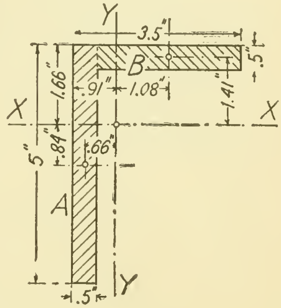


Fig. 390

222. Use of Transfer Formula. Let it be required to find the product of inertia of the angle section shown in Fig. 390, referred to the axes XX and YY. Dividing the area into rectangles as shown, and using equation (1), §221, for each rectangle, we have

$$\begin{aligned} &\text{For } A: \quad K_1 = 0 + (-0.66)(-0.84)(2.5) = 1.386 \\ &\text{For } B: \quad K_1 = 0 + (+1.08)(+1.41)(1.5) = 2.284 \\ &\text{For the whole angle.} \qquad K_1 = 3.670''^4 \end{aligned}$$

PROBLEMS

1. Find the product of inertia of the section shown in Fig. 249, referred to H and V axes through the center of gravity.
2. Repeat Problem 1 using Fig. 126
3. Repeat Problem 1 using Fig. 129.
4. Repeat Problem 1 using Fig. 119.
5. Repeat Problem 1 using Fig. 130.
6. In Problem 4, let the axes intersect at n.

223. Transfer between Inclined Axes.—Moment of Inertia and Product of Inertia. Let Fig. 391A represent any area. Let I_x and I_y be the moments of inertia with respect to the X and Y axes, respectively. Let K be the product of inertia with respect to the X and Y axes, and let I_x, I_y, and K be known. Let it be required to find I_{x_1}, I_{y_1}, and K_1, all of which are referred to the X_1 and Y_1 axes and are unknown. The origin o is common to the two sets of axes.

Let dA be an elementary area whose coordinates with respect

to the two sets of axes are shown. The relations between the two sets of coordinates (see Fig. 391B) are

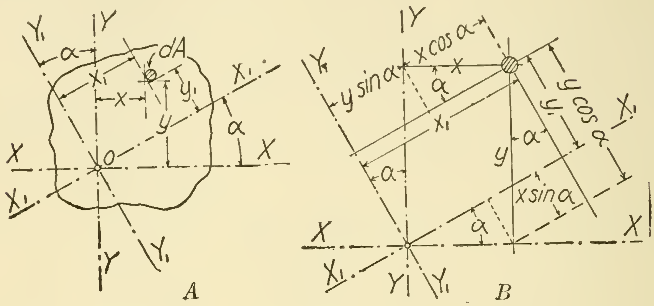


FIG. 391

(2) $$x_1 = x \cos \alpha + y \sin \alpha,$$

(3) $$y_1 = y \cos \alpha - x \sin \alpha.$$

Furthermore, by definition (§§ 107 and 219), we have

$$I_{x_1} = \int_A y_1^2 dA,$$

$$I_{y_1} = \int_A x_1^2 dA,$$

$$K_1 = \int_A x_1 y_1 dA.$$

Substituting the values for x_1 and y_1 derived in (2) and (3), we find

$$I_{x_1} = \int_A (y \cos \alpha - x \sin \alpha)^2 dA$$

$$= \cos^2 \alpha \int_A y^2 dA - 2 \cos \alpha \sin \alpha \int_A xy\, dA + \sin^2 \alpha \int_A x^2 dA,$$

or

(4) $$I_{x_1} = I_x \cos^2 \alpha + I_y \sin^2 \alpha - 2K \cos \alpha \sin \alpha.$$

By the same method we could show that

(5) $$I_{y_1} = I_y \cos^2 \alpha + I_x \sin^2 \alpha + 2K \cos \alpha \sin \alpha,$$

and that

(6) $$^*K_1 = \cos\alpha\sin\alpha(I_x - I_y) + (\cos^2\alpha - \sin^2\alpha)K.$$

By use of the trigonometric relations

$$\cos^2\alpha = \frac{1+\cos 2\alpha}{2}, \qquad \sin^2\alpha = \frac{1-\cos 2\alpha}{2},$$

$$2\cos\alpha\sin\alpha = \sin 2\alpha, \qquad \cos^2\alpha - \sin^2\alpha = \cos 2\alpha,$$

equations (4) to (6) can be transformed to the following forms which are sometimes more convenient:

(7) $$I_{x_1} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\alpha - K\sin 2\alpha,$$

(8) $$I_{y_1} = \frac{I_y + I_x}{2} - \frac{I_x - I_y}{2}\cos 2\alpha + K\sin 2\alpha,$$

(9) $$K_1 = \frac{I_x - I_y}{2}\sin 2\alpha + K\cos 2\alpha.$$

From (4) and (5), or from (7) and (8), by addition, it follows that

(10) $$I_{x_1} + I_{y_1} = I_x + I_y.$$ √

This statement is independent of the value of α.

If the moment of inertia and the product of inertia of a given area referred to a given pair of rectangular axes are known, these equations will enable us to find the moment and product of inertia with respect to any other axes having the same origin as the first. Thus by successive use of the equations in §§ 113, 221, and 223, we can transfer the axes for either I or K, at will. In the above discussion α is positive when measured from XX in a counter-clockwise direction.

PROBLEMS

1. Find the moment of inertia of a 10″ × 25 lb. I beam about each of a pair of rectangular gravity axes inclined at 30° to the plane of the web.

* For the *special case* when the transfer is being made from the principal axes (§ 224), we know (§ 227) that $K = 0$. Therefore, *for that case*, the above equations (4) and (5) become

(11) $$I_{x1} = I_x\cos^2\alpha + I_y\sin^2\alpha,$$

(12) $$I_{y1} = I_y\cos^2\alpha + I_x\sin^2\alpha.$$

2. Find the moment of inertia of a 12″ × 16″ rectangle about each of a pair of rectangular axes passing through a corner of the figure and making angles of 45° with its sides.
3. (*a*) Find the moment of inertia of the trapezoid, Fig. 195, referred to an axis passing through the lower left corner and making an angle of 30° with *EF*. (*b*) Repeat using an angle of − 30°.
4. Determine the moment of inertia of the section, Fig. 197*A*, referred to an axis passing through the intersection of *1–1* and *2–2* (center of gravity), and making an angle of 20° with *1–1*.

224. Principal Axes. Equations (4) and (5), § 223, show that the moment of inertia varies with the angle α, Fig. 391*A*. Evidently some value of α will render I_{x_1} a maximum. Let us assume that for a given area and a given origin, I_x and I_y are known, and that we wish to find the position of the axes, having the same origin as the first, and for one of which I is a maximum. The required position can be expressed in terms of the angle between the axes, i.e., in terms of α, Fig. 391. Equation (7), § 223, gives

$$I_{x_1} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\alpha - K \sin 2\alpha.$$

Using the ordinary process of the calculus to determine a maximum, we find

$$\frac{dI_{x_1}}{d\alpha} = -2\left(\frac{I_x - I_y}{2}\right) \sin 2\alpha - 2K \cos 2\alpha$$

$$(13) \qquad = -(I_x - I_y) \sin 2\alpha - 2K \cos 2\alpha.$$

Putting this equal to zero, we have

$$(I_y - I_x) \sin 2\alpha = 2K \cos 2\alpha,$$

whence

$$(14) \qquad \tan 2\alpha = \frac{2K}{I_y - I_x}.$$

When α satisfies equation (14), I_{x_1} is a maximum or a minimum. It should be noted that every value of the tangent corresponds to two angles differing by 180°. Thus the above equation will result in two values of 2α and these give rise to two values of α which differ by 90°. One of these values corresponds to the

maximum value of I_{x_1}, the other to the minimum value, and, at the same times, I_{y_1} is a minimum or a maximum. From the above, it becomes clear that, with *any* given point as an origin, there is a pair of rectangular axes about one of which I is a maximum and about the other it is a minimum. Such axes are called the ***principal axes.*** The origin *most frequently used* is the center of gravity of the section, since the neutral surface of a beam passes through the center of gravity.* Unless otherwise specially noted, the term *principal axes* will hereafter imply an origin at the center of gravity of the section.

In order to designate principal axes clearly, we shall hereafter use X', Y' to designate the principal axes and $I_{x'}$, $I_{y'}$, and K' to designate the moments of inertia and the product of inertia when referred to principal axes. In this case, as in § 223, a positive value of α indicates that the angle is measured in a *counterclockwise* direction *from* the axes about which I is known.

225. Principal Axes for an Angle. Let it be required to find the principal axes, with the center of gravity as the origin, for the angle in Fig. 392. From a handbook, $I_x = 10.0$, $I_y = 4.0$, and from § 222, $K = 3.67$. From equation (14), § 224,

$$\tan 2\alpha = \frac{2(3.67)}{4 - 10} = -1.223$$

From a table of trigonometric functions, we have, approximately,

$$2\alpha = -50° 50', \qquad \text{or} \qquad 129° 10',$$
$$\alpha = -25° 25', \qquad \text{or} \qquad 64° 35'.$$

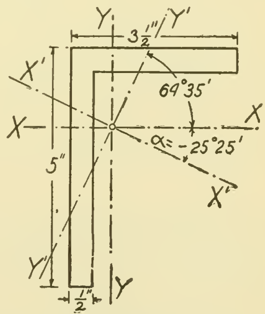


Fig. 392

These axes are shown dotted in the figure, and are principal axes passing through the center of gravity of the angle section. The moments of inertia with respect to these axes may now be found by use of equations (4) and (5), § 223, using the value $\alpha = -25° 25'$. Equation (4) will give the moment of inertia about the axis $X'X'$. The value is found to be $11.75''^4$, and

* For certain exceptions see §§ 208–210.

this is the greatest I for any axis through the center of gravity. Using the same value of α, equation (5) will give the moment of inertia about the $Y'Y'$ axis, which is found to be $2.25''^4$. This is the least possible value.

226. Least Radius of Gyration. Because of its importance in the design of columns (§§ 182 and 186—3) the least radius of gyration of an area is often required. It is easily derived as soon as the principal axes of the section have been determined, as outlined in § 224.

For the angle section in Fig. 392, we have the principal moments of inertia (see § 225) as $11.75''^4$ and $2.25''^4$, while the area is 4.0 sq. in. The least r will then be $\sqrt{2.25/4} = 0.75''$, figured about the axis $Y'Y'$.

227. Axes of Symmetry as Principal Axes. In Fig. 393, let the moments of inertia I_x and I_y and the product of inertia K with respect to the XX and YY axes be known, and let the $X'X'$ and $Y'Y'$ axes be the principal axes. Then from equation (9), § 223, we have

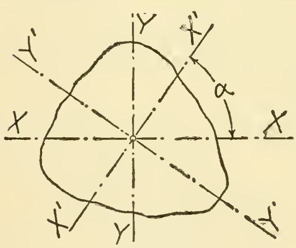


Fig. 393

$$(15) \quad 2K' = (I_x - I_y) \sin 2\alpha + 2K \cos 2\alpha,$$

but this value of K' will be zero, by (13), since the axes are principal axes. This shows that when the axes are principal axes, $K' = 0$. But we know that when one of the axes is an axis of symmetry of the figure, $K = 0$ (§ 220A). It follows that an axis of symmetry is a principal axis.

228. General Propositions.—Moment of Inertia and Product of Inertia. In addition to the general summary on moment of inertia given in § 116, we can now add the following facts.

(10) Moment of inertia may be transferred from one axis to another making an angle with the first (§ 223).

(11) For any point, taken as the origin of coordinates, there is one set of rectangular axes (principal axes) about one of which I is the greatest and about the other of which I is least. These

are the principal moments of inertia with respect to this point (§ 224).

(12) Any axis of symmetry is a principal axis (§ 227).

(13) An inspection of equations (11) and (12) (§ 223) will show that if the principal moments of inertia are equal the moments of inertia about *all* axes passing through the origin are equal.

(14) Since, from equation (10), § 223, $I_{x_1} + I_{y_1} = I_x + I_y$, it follows that the sum of the principal moments of inertia is equal to the sum of the moments of inertia with respect to any other pair of rectangular axes having the same origin. This is sometimes useful in checking results of transformations of axes.

The following principles refer to product of inertia.

(1) Product of inertia is zero for any figure symmetric about an axis of coordinates, § 220A.

(2) When K is zero, the axes are principal axes, § 227.

In more extended texts, graphic methods of dealing with moments of inertia are given. They are valuable time savers provided the general principles as here given have been mastered.

PROBLEMS

1. Determine the principal axes, maximum moment of inertia, and least radius of gyration of an $8'' \times 6'' \times 1''$ angle. Let the axes intersect at the center of gravity.
2. Repeat Problem 1 using a Z bar 4″ deep with flanges 3″ wide and a thickness throughout of $\frac{1}{2}''$.
3. Repeat Problem 1 using the section in Fig. 130.

229. Components of the Bending Moment. In §§ 217 and 218, we have examined the question of unsymmetric bending by way of the components of the *resisting moment.* We will now resume the discussion by considering the components of the *bending moment.* It is worth while first to redefine unsymmetric bending in the light of the principles developed in §§ 219–228. As defined in § 131, unsymmetric bending occurs when the forces producing bending lie in a plane which does not coincide with an axis of symmetry of the beam itself. It is easily deducible from §§ 117–228 that *in so far as the theory of bending is concerned* principal

axes might well be considered as equivalent to axes of symmetry. This is demonstrated in § 231. It is then evident that we may recast the previous definition to read as follows. Unsymmetric bending occurs when the forces producing bending do not lie in a principal axial plane of the member.

Obviously this definition admits many different arrangements of the forces producing unsymmetric bending, but in each case it is possible to reduce the bending moment to component moments which lie in or are parallel to the principal axial planes.

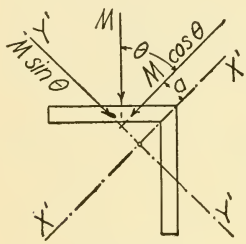


Fig. 394

A. Transverse Loading. When a beam is loaded as shown in Fig. 333, the load P produces a bending moment equal to $PL/4$. This moment can be resolved into a moment of $(PL/4) \sin \theta$ acting parallel to one of the principal axes and another, $(PL/4) \cos \theta$, acting parallel to the other principal axis. This case can then be worked out completely as in §§ 195 or 232B. If the transverse loading is not in a plane passing through the center of gravity of a section, as shown by M in Fig. 394, the moment can be resolved into one moment which is in the plane of a principal axis ($M \sin \theta$) and another which is parallel to the other principal axis ($M \cos \theta$). This latter momen produces a twisting moment ($aM \cos \theta$) in addition to the bending moments. Twisting moments, in general, produce shearing st esses, as explained in § 246. When the twisting moment is not excessive, this case can be treated like the previous one. An analysis of the case which takes account of the twisting moment on the irregular sections in common use cannot be attempted in this text.

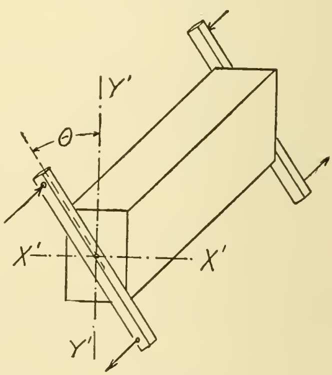


Fig. 395

B. Terminal Couples. In Fig. 395 is shown a bar acted upon by terminal couples, not in the plane of a principal axis. Let the moment produced by the couples be M. Then the effect of this moment can be reduced to a moment of $M \cos \theta$, acting about the X' plane, and one of $M \sin \theta$, about the Y' plane. A case of this sort is worked out in § 232C.

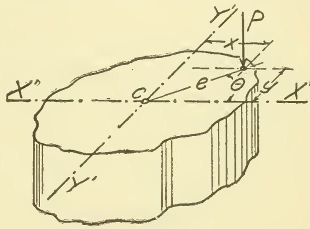
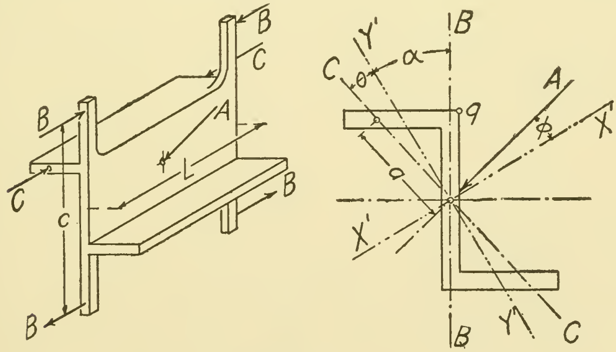


Fig. 396

C. Eccentric Direct Loading. The effect of an eccentric load acting along the length of a bar or short block is essentially the same as that of a direct concentric load plus a terminal couple, as shown in §§ 192–194, for the case of symmetric bending. For the case of unsymmetric bending, let Fig. 396 represent a short block with its principal axes X' and Y'. The effect of the load P can be resolved into a direct force P, acting at c; and a moment $M = Pe$. But the moment M can be further resolved into two moments: Px tending to produce rotation about Y', and Py tending to produce rotation about X'. But $x = e \cos \theta$



A Fig. 397 *B*

and $y = e \sin \theta$. Therefore the effects produced by the load P are as follows. (1) A direct force, P, acting at c. (2) A moment, $Px = Pe \cos \theta$, producing rotation about Y'. (3) A moment, $Py = Pe \sin \theta$, producing rotation about X'. This case is discussed in full in § 232A.

D. Combinations. Let Fig. 397A represent a Z bar acted upon by three loads A, B, and C, corresponding to the above cases, and let Fig. 397B represent a cross section of the bar. The planes of the external moments are shown by dotted lines, and $X'X'$, $Y'Y'$ are the principal axes.

The direct load C produces a direct compressive stress C, a moment $Ca \cos\theta$ about the X' plane, and a moment $Ca \sin\theta$ about the Y' plane. The load A, at the center of the span, produces a bending moment $AL/4$. Its component effects are $(AL/4)\cos\phi$ about the Y' plane and $(AL/4)\sin\phi$ in the X' plane. The terminal couples produce a moment Bc which may be resolved into $Bc\cos\alpha$ about the X' plane and $Bc\sin\alpha$ about the Y' plane.

If we call a moment producing compression on the point q positive, then the total resultant moment about the X' plane will be

$$Ca\cos\theta + \frac{AL}{4}\sin\phi + Bc\cos\alpha,$$

and, about the Y' plane, the resultant moment will be

$$-Ca\sin\theta + \frac{AL}{4}\cos\phi + Bc\sin\alpha,$$

to which must be added the direct effect of the load C.

230. Unit Stress Due to Unsymmetric Bending. Let Fig. 398 represent the cross section of a bar acted upon by a bending moment in the plane ZZ. This moment may be produced in any of the ways described in § 229. It is required to determine the unit stress at *any* point as (p) on the section, due to the bending moment.

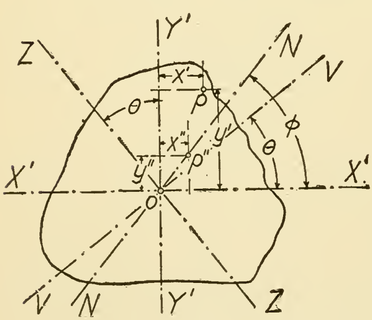


Fig. 398

The following solution depends upon resolving the moment into components which act about the planes of the *principal axes* of

the section. When this is done, each component moment produces symmetric bending (as shown in § 229) and the unit stress *due to it* can be determined by the *ordinary* theory of bending. The effects of the component moments can then be combined in the usual manner.

In Fig. 398, $X'X'$ and $Y'Y'$ are principal axes; o is the center of gravity of the section; ZZ is the plane of the external moment, whose amount is M, and which is so directed as to produce compression on the part of the section above $X'X'$; NN represents the neutral surface whose position is as yet undetermined; and VV is perpendicular to the plane of the moment ZZ.

As was shown in § 229, the resultant moment can be resolved into one component ($M_{x'}$), acting about the X' axis, and another ($M_{y'}$), acting about the Y' axis. These moments are taken as positive when they produce compression on the point p and negative when they produce tension. Their amounts are

$$M_{x'} = M \cos \theta, \qquad \text{and} \qquad M_{y'} = - M \sin \theta. \tag{16}$$

As noted above, the unit stresses due to these moments can be found by the ordinary theory of bending and combined in the usual way, as follows. Let $s_{x'}$ be the unit stress at p due to $M_{x'}$, and $s_{y'}$ be that due to $M_{y'}$. Let $I_{x'}$ and $I_{y'}$ be the moments of inertia about the *principal* axes. Then we have

$$M_{x'} = \frac{s_{x'} I_{x'}}{y'}, \qquad \text{and} \qquad M_{y'} = \frac{s_{y'} I_{y'}}{x'};$$

$$s_{x'} = \frac{M_{x'} y'}{I_{x'}}, \qquad \text{and} \qquad s_{y'} = \frac{M_{y'} x'}{I_{y'}}.$$

Substituting the values of $M_{x'}$ and $M_{y'}$ from equation (16), we find

$$s_{x'} = \frac{M \cos \theta y'}{I_{x'}}, \qquad \text{and} \qquad s_{y'} = - \frac{M \sin \theta x'}{I_{y'}},$$

and the total stress at the point p is

$$s = s_{x'} + s_{y'} = M\left(\frac{y' \cos \theta}{I_{x'}} - \frac{x' \sin \theta}{I_{y'}}\right). \tag{17}$$

This equation gives the unit stress at the point p due to the moment M, as illustrated in Fig. 398. If the unit stress so determined is positive, it is compression. The minus sign indicates tension. In order to make equation (17) apply to any position of the point p, it may be written in the form

$$(18) \qquad s = M\left(\pm\frac{y' \cos\theta}{I_{x'}} \pm \frac{x' \sin\theta}{I_{y'}}\right).$$

In this equation the notation is as follows.

s = the unit stress at any point p, due to unsymmetrical bending. When s has a positive value the unit stress is compression, and when s is negative the unit stress is tension.

M = the bending moment (usually in pound-inches).

x' and y' = the coordinates of the point p, *referred to the principal axes.*

$I_{x'}$ and $I_{y'}$ = the *principal* moments of inertia of the section.

θ = the angle between the X' axis and the trace of a plane perpendicular to the plane of the bending moment. This angle may always be considered as positive, *provided* the $\pm$ signs are arranged to take account of the tendency of the component moments to produce compression ($+$) or tension ($-$) on the point at which the unit stress is being computed. This requires that the $+$ or $-$ sign used in front of each term in the parenthesis of equation (18) shall be separately determined on the basis of tensile or compressive stress.

Equation (18) above gives the unit stress at *any* point on the cross section. But it is usually required to find the unit stress at the fiber most remote from the neutral surface. So far no means of locating the neutral surface has been determined. This will be developed in § 231.

231. Position of the Neutral Surface. When it is desired to locate the neutral surface, we make use of the fact that the unit stress at the neutral surface is zero. Thus in Fig. 398, let NN represent the neutral surface, and let p'', whose coordinates are (x'', y'') be any point on it. Then, from equation (17), § 230, the unit stress at p'' is equal to

$$s = 0 = M\left(\frac{y'' \cos\theta}{I_{x'}} - \frac{x'' \sin\theta}{I_{y'}}\right).$$

Solving this equation, we obtain

$$(19) \qquad y'' = x'' \frac{I_{x'}}{I_{y'}} \tan\theta.$$

From the relations shown in Fig. 398, $y''/x'' = \tan\phi$. Dividing equation (19) by x'', we get

$$(20) \qquad \tan\phi = \frac{I_{x'}}{I_{y'}} \tan\theta.$$

Here θ and ϕ are measured from the principal X' axis and are positive when measured counterclockwise.

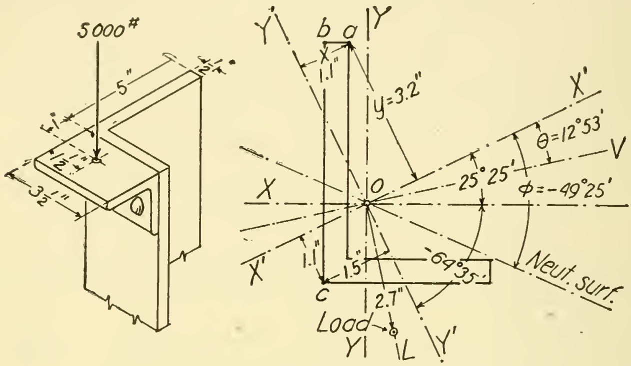
Equation (20) gives the position of the neutral surface when the principal axes are known and the plane of the moment is fixed. It may further be noted that, in the above expression, when θ has the value zero, ϕ also has the value zero. That is to say, when the plane of the bending moment coincides with a principal axis, the neutral surface is perpendicular to the plane of the bending moment. This is the condition peculiar to what we have heretofore considered as symmetric bending. We may therefore conclude (as forecast in § 229) that, *in so far as bending is concerned,* principal axes are equivalent to axes of symmetry.

232. Problems.—Unsymmetric Bending. A. Eccentric Direct Load (see § 229C). Let the $5'' \times 3\frac{1}{2}'' \times \frac{1}{2}''$ angle shown in Fig. 399*A* carry an eccentric load as shown, and let it be required to find the maximum unit stress on the section of the angle. The cross section is shown (enlarged) in Fig. 399*B*.

The unit stress at any point is made up of a stress P/A due to direct compression, and a stress due to unsymmetric bending. The stress due to this bending can be determined as follows. From a handbook of steel sections, we find $I_x = 10$; $I_y = 4$; area of section $= 4$; and the location of the center of gravity, as shown at o in the drawing. From § 222, we see that the

product of inertia referred to axes X and Y is $-3.67''^4$.* Then, from § 225, we find that the principal axes are inclined at angles of 25° 25′ and − 64° 35′ to the axis XX.*

Next, using equations (4) and (5), § 223, we find that the moment of inertia about X' is 11.75, and that about Y' is 2.25.



A Fig. 399 *B*

We might now proceed, by use of equation (18), § 230, to find the unit stress at any point on the cross section of the angle. But we wish to find the *maximum* unit stress on the section. This could be done by making trial computations for several likely points such as a, b, and c, and then selecting the largest value. But a direct solution is possible if we first locate the neutral surface.

The trace of the moment plane is shown by Lo; and oV is perpendicular to Lo. By measurement, the angle θ between X' and oV is found to be − 12° 53′, and by using equation (20), § 231, the angle ϕ between the X' axis and the neutral surface is found to be − 49° 25′, as shown. When the neutral surface is drawn, it becomes evident that points a and c are the outermost fibers on the tension and compression side of the neutral surface. Their distances from oX' and oY' may now be scaled off, and we are ready to solve for the unit stress by use of equation (18), § 230. This gives

* Figure 399 shows the section in a position reversed from that shown in Fig. 392. Therefore, the plus and minus signs applying to K and to the various angles are reversed.

$$s_a = 5{,}000(2.7)\left(\frac{3.2(0.9748)}{11.75} + \frac{1.1(0.2231)}{2.25}\right)$$
$$= 5{,}058 \quad \text{lbs. per sq. in. tension;}$$
$$s_c = 5{,}000(2.7)\left(\frac{1.1(0.9748)}{11.75} + \frac{1.5(0.2231)}{2.25}\right)$$
$$= 3{,}240 \quad \text{lbs. per sq. in., compression.}$$

These are the stresses due to bending moment only. The stress due to the direct load is

$$s = 5{,}000/4 = 1{,}250 \quad \text{lbs. per sq. in., compression.}$$

Then the total resultant stresses are

$$s_a = 5{,}060 - 1{,}250 = 3{,}810 \quad \text{lbs. per sq. in., tension, at } A;$$
$$s_c = 3{,}240 + 1{,}250 = 4{,}490 \quad \text{lbs. per sq. in., compression, at } C.$$

The solution given in § 192A is a special case of this solution in which $\theta = 0$.

B. Transverse Loading.—Inclined Forces. Let Fig. 400 represent the cross section of a beam 10′ 0″ long, carrying a uniformly distributed load of 3,000 lbs. which acts in the plane indicated by the arrow W. This section is composed of one 3″ × 3″ × ½″ angle and one 5″ × 3″ × ½″ angle. The bottom flanges are inclined at 30° to the horizontal. Let it be required to find the greatest unit stress on the section. From a handbook we can get the centers of gravity and the moments of inertia of the sections, referred to axes parallel to the legs of the angles.

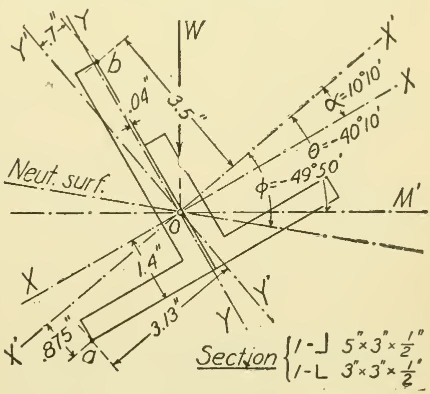


Fig. 400

Then, by § 52, we can locate the center of gravity of the

section as shown at o. Also the moments of inertia of the combination with respect to the axes marked X and Y can be found from § 114; they are $I_x = 12.77$; $I_y = 9.27$. Then, by § 221, $K = -0.649$.

Next, by § 224, the location of the principal axis is determined, the angle α being 10° 10′, as shown on the drawing. Next, by use of equations (4) and (5), or (7) and (8), § 223, the moments of inertia about these axes are found. $I_{x'} = 12.86$; $I_{y'} = 9.15$. By measurement θ is found to be $-$ 40° 10′. We can now locate the neutral surface by use of equation (20), § 231. The angle ϕ is found to be $-$ 49° 50′, as shown. From a full-sized drawing, the distances from the points a and b to the principal axes are scaled. Lastly, from equation (18), § 230, the unit stress at the point a is found to be 12,250 lbs. per sq. in. tension; and at the point b it is found to be 11,600 lbs. per sq. in. compression. The solution given in § 195 is a special case of the one here given.

C. Bending by Terminal Couples. Let Fig. 401 represent the cross section of a Z bar, bent by terminal couples (similar to Fig. 382), acting in the vertical plane Y, and having a moment of 15,000 lb. ins. so directed that the stress will be compressive on the point a. Required the maximum unit stress on the section. From the handbook, we get $I_x = 19.2$, $I_y = 9.1$; and by use of § 221, $K = 10.07$. From § 224, the principal axes are found and are located on the drawing as shown: $\alpha = -$ 31° 41′. From § 223, the moments of inertia with respect to the principal axes are found to be $I_{x'} = 25.41$, $I_{y'} = 2.88$.

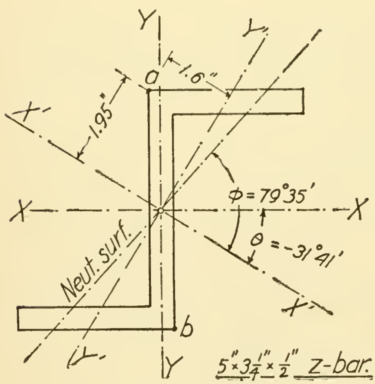


Fig. 401

It seems obvious that the most stressed fibers will be those marked a and b. However, if it is desired that the neutral surface be known, we may use equation (20), § 231. In this

case the angle θ is $-31° 41'$ and ϕ is found to be $79° 35'$, as shown. From a full-sized drawing of the section, the distances of the points a and b from the principal axes are determined. Then, using equation (18), § 230, the unit stress at a or b is found to be 5,350 lbs. per sq. in.

If treated by the *ordinary* theory of bending, the unit stress in the above case would be found to be 1,960 lbs. per sq. in. This shows how the ordinary theory, if wrongly applied to a case of unsymmetric bending, may give results far too small and may therefore give a false sense of security.

PROBLEMS

1. The Tee section shown in Fig. 402 is used to carry a vertical load of 1,000 lbs., concentrated at the center of a span of 8′ 0″. What is the maximum unit stress?
2. In Problem 1, what is the safe load if the allowable unit stress is 13,000 lbs. per sq. in.?
3. A 5″ × 3½″ × ½″ angle is placed with the long leg vertical. It acts as a beam whose span is 10′ 0″. The load is 1,800 lbs., uniformly distributed and vertical. Find the maximum unit stress.

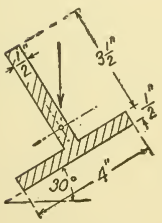


FIG. 402

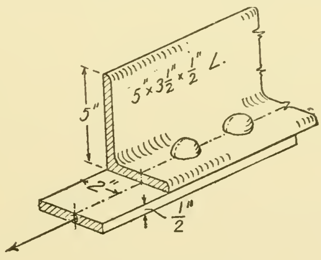


FIG. 403

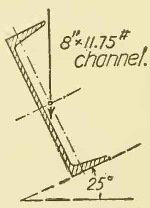


FIG. 404

4. A 5″ × 3½″ × ½″ angle is used as a tie member in a truss. The total tension is 12,000 lbs. The connection is made by means of a gusset plate as shown in Fig. 403. What is the maximum unit stress?
5. Let the angle in Fig. 403 carry a compressive stress of 8,000 lbs., the length being 5′ 6″. Is it safe?
6. An 8″ × 11¼ lb. channel is used as a purlin to carry a uniformly distributed load of 3,000 lbs. on a span of 10 ft. The load is in a vertical plane while the channel is placed as shown in Fig. 404. What is the maximum unit stress?

7. The section shown in Fig. 130 is used as a beam to span 20′ 0″, carrying a uniformly distributed load which acts in a vertical plane. What is the safe load?
8. The section shown in Fig. 400 is used as a beam 12′ 0″ long. The 5″ leg is set vertical and the beam carries a vertical load uniformly distributed. What is the safe load?
9. What is the safe resisting moment of a $6'' \times 3\frac{1}{2}'' \times \frac{3}{4}''$ Zee bar when bent by terminal couples that lie in the center plane of the web of the bar?

CHAPTER XXIII

PROBLEMS INVOLVING WORK

233. Introduction. As pointed out in § 6, the problems of structural engineering are chiefly those of statics. However, there are some problems that, for their solution, depend upon principles outside the field of pure statics. These problems are relatively rare, but when they do arise they become highly important. In this chapter no attempt will be made to treat them in detail. The solutions in general become quite complex. They are available in more extended texts. But it seems worth while to discuss here a few of the simplest cases, and to point out the underlying principles of the more complex ones. We shall notice two classes of problems.

(1) *Loads not static.* In most structures the entire load, or at least a very large percentage of the entire load, is static. But in certain cases, such as bridges, the moving loads may be of such importance as to call for special attention.

(2) *Structure statically indeterminate.* When a structure is such that the stresses in it are statically indeterminate (§§ 40 and 163), the problem can sometimes be solved by a consideration of the motions set up as the structure deforms under loading.

234. Basic Ideas. When a body is in motion, its motion is the resultant of the actions of certain forces and resistances. If the forces are greater, the velocity is increasing. If the resistances are greater, the velocity is decreasing. Again, if the motion is the result of an original impulse which is not sustained, the resistances ultimately overcome the tendency to move and the body comes to rest. Here we have a set of ideas requiring some specific definitions.

(1) *Work.* When a force acts through a distance it is said to perform a certain amount of work. Thus in Fig. 405 if the drum A is one third the diameter of the drum B, the force F, acting

through the distance ab, will move the weight W through $a'b' = \frac{1}{3}ab$. If the machine is frictionless, and if $F = W/3$, the velocity imparted to the weight will be uniform. The force F has performed an amount of work which is measured by the force (say 10 lbs.) times the distance (say 6′ 0″). The work performed is then 60 ft. lbs. At the same time the work expended in raising the weight has been 30 lbs. × 2′ = 60 ft. lbs. If the machine is not frictionless, the force F must be greater than $W/3$ if motion is to result. Then $F \times ab$ must be greater than $W \times a'b'$ by the amount needed to overcome friction. If the machine is frictionless and $F > W/3$ the motion will be accelerated and if $F < W/3$ the motion will be reversed.

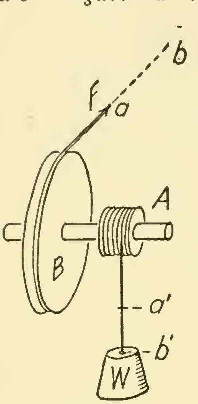


Fig. 405

Work can be defined as force multiplied by the distance through which it acts. It is ordinarily expressed in foot-pounds.

(2) *Energy* is the capacity for doing work. This capacity may be the result of velocity or of position. Thus a projectile, having a certain vertical velocity imparted to it, has the capacity of overcoming the resistances of the air and of gravitation. It is said to possess a certain energy. By virtue of this energy it rises, performing a certain amount of work. When the amount of work done against the resistances is equal to the work done in imparting the original velocity, the projectile comes to rest. It now has the capacity of overcoming the air resistance in falling and of exerting forces in its impact with the ground. Thus the projectile starts out with a certain energy due to velocity (called *kinetic energy*) which, at the *top* of its rise, has been partly expended in heat due to overcoming air resistance, and partly transformed into another sort of energy (*potential energy*) which is due to the position of the body. If the air resistances be neglected, the kinetic energy at the start and the potential energy at the top of the flight are equal; and, at the bottom of the return flight, the kinetic energy is again equal to

the original. The doctrine of the *conservation of energy* is one of the fundamental hypotheses of physics. According to this doctrine, energy is indestructible. The doing of work is then but the transforming of one kind of energy into another: such as kinetic, potential, or heat, electricity, etc.

(3) *Power* is the rate at which work is done. Thus if a force of 100 lbs. moves through 5′ 0″ in one second, the *power expended* is 500 ft. lbs. per second. The commonly used unit called a ***horse power*** is 550 ft. lbs. per second, or 33,000 ft. lbs. per minute.

PROBLEMS

1. A body weighing 75 lbs. is pulled upward along a smooth plane which is inclined at 30° to the horizontal. How much work has been done when the body has traveled 15′ along the plane?
2. An elevator car with its load weighs 2,500 lbs. The counter weight is 1,600 lbs. What is the horse power required to produce a uniform velocity of 800 ft. per minute?

235. Suddenly Applied Loads. Let Fig. 406 represent a flanged rod and imagine a ring R to be *supported* so that it is in contact with *but not resting on* the flange of the rod at A. Let the weight of the ring be P.

If the support under the ring is *gradually* withdrawn, the load will be gradually transferred to the flange and the flange will gradually move from A to some position B where it will come to rest. Throughout this operation the stress in the rod is increasing from zero up to the final value of P.

If the support is *suddenly* (instantaneously) withdrawn from under the ring, the load, when it reaches B, will have a definite kinetic energy which will carry it on to some still lower position C. When it reaches C, the work done by the moving load has all been absorbed in elongating the rod, and enough energy has been stored within the rod to pull the load nearly back to its first position. These oscillations will continue until all of the work done by the load has been dissipated in the form of heat, etc., or in storing energy in the rod.

If we assume that the stress in the rod increases uniformly from zero (when the flange is at A) to some amount P' (when

the flange is at C), the average stress in the rod has been $P'/2$ and this has been acting through the distance d; hence the energy stored in the rod is $P'd/2$.

The load possessed no kinetic energy in the beginning; nor does it possess any at the lowest point C. Whatever work it has done has been done in extending the rod. But the work done by the load is evidently Pd. Hence

$$(1) \qquad \frac{P'}{2}d = Pd \qquad \text{and} \qquad P' = 2P;$$

that is, the stress due to a suddenly applied load is twice as great as that due to the same load when applied gradually.

236. Impact. If the load in Fig. 406 is raised to D and then allowed to drop on the flange of the rod, the stresses developed are greater than before. The work done in this case will be $P(d + e)$ and equation (1), § 235, becomes

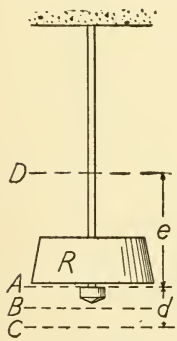


Fig. 406

$$\frac{P'}{2}d = P(d + e),$$

$$P' = 2P\frac{(d + e)}{d}.$$

When e is large in comparison with d, i.e., when the moving load has a considerable velocity when it reaches the ring, the stresses produced may be much greater than those due to static loads.

In more extended texts formulas are developed to evaluate the effect of impact on beams and for other cases, but in structural practice they are not much used, since it is extremely rare to encounter a case in which known loads, traveling at known speeds, produce direct impact on the structure. The cases which actually arise are rather those of suddenly applied loads, as when a train moves rapidly over a bridge, or of loads falling through small and indeterminate distances, as when a roughness in the road bed creates actual impact from wheel loads.

General practice favors making a single allowance for impact which varies with the probable conditions under which the structure must function. This allowance may take one of the following forms.

(1) *Loading.* The estimated live loads are increased by a definite percentage over their known static amounts.

(2) *Stresses.* The working unit stresses, used for static loading, are decreased by a definite percentage.

(3) *Factors of Safety.* The factors of safety used for static cases are increased by a definite percentage.

Evidently the net result of any one of these allowances is to put more material into the structure to counter-balance the effect of the impact. These allowances vary widely for different conditions and the amount to be allowed is a matter of individual judgment.

237. External and Internal Work. If any piece of material is acted upon by a force of P lbs., and is deformed to the amount of e'', then work has been performed by the force in causing the deformation of the piece. During the period of deformation the force applied increases from zero, at the beginning, to its full amount P, when the full deformation is accomplished. If the applied force increases uniformly, the *average* force is $P/2$. This force acts through the distance e and the work done is $Pe/2$ in. lbs. This is known as the *external work* applied to the body. At the same time there has been stored within the piece an energy equal to $Pe/2$ in. lbs. (§ 235), which is sometimes called internal work. The fundamental principle here involved is sometimes stated in the form

$$\text{External Work} = \text{Internal Work.}$$

In more advanced texts, it is proved that this relation holds good not only for single members, but also for structures composed of many members, such as trusses or beams, or even for structures in which bending and direct stresses are combined.

238. Principle of Least Work. In the case of statically indeterminate structures (§ 40) the determination of the stresses

due to given loads almost invariably becomes quite complex. In general, the solutions for such cases depend on the elastic properties of the structure. In the case of restrained bending (Chap. XVIII) we have already encountered one such type of solution. A different type of problem has to do with framed structures containing *redundant* members.

If a framed structure contains more members than are necessary to its stability, it is said to contain ***redundant*** members. Thus in Fig. 63, the horizontal member could be eliminated and the truss would still function to carry its loads. In Fig. 65, the vertical bar is not a necessary part of the structure. In each of these cases, we have seen (§ 40) that the principles of statics will not yield a solution. In such cases the principle of least work can be used. No proof of this idea will be given here since it is somewhat involved. The proof can be found in more advanced texts. The general idea is so readily acceptable that most people are inclined to view it as axiomatic. It can be stated as follows:

When a structure contains redundant members or reactions, the internal stresses are so distributed throughout the structure that the energy stored in the members is a minimum.

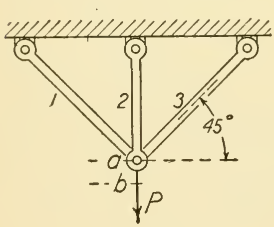


Fig. 407

Figure 407 illustrates such a case. The lengths of the bars before the load is applied are such that the position of the pin joining them is a. After the load is applied the pin is lowered, say to b, and stresses are set up in the bars. Let these stresses be called T_1, T_2, and T_3, and the elongations of the bars be called e_1, e_2, and e_3. Similarly A, L, and E, with appropriate subscripts, will represent the area, length, and modulus of elasticity of the various bars. Then the internal work, by § 237 (or the energy stored in the bars), is

$$\frac{T_1e_1}{2}+\frac{T_2e_2}{2}+\frac{T_3e_3}{2}.$$

This total work varies as the point b changes its position. The

principle of least work asserts that the position of b will be such as to render the total work a minimum.

The method of solution, using this principle, is as follows. From symmetry, we can say that $T_1 = T_3$. From statics, we know that the vertical components of the stresses in all the bars will be equal to P; or

$$2(0.707T_1) + T_2 = P,$$

whence

(2) $$T_1 = \frac{P - T_2}{1.414}.$$

The deformation of bar 1 is

$$e_1 = \frac{T_1L_1}{A_1E_1},$$

and the energy stored in the bar is then $T_1^2L_1/(2A_1E_1)$. For bar 2 the stored energy is $T_2^2L_2/(2A_2E_2)$. Then the total energy stored in the three bars will be

$$R = \frac{2(T_1^2L_1)}{2A_1E_1} + \frac{T_2^2L_2}{2A_2E_2}.$$

By the principle of least work, this must be a minimum. Treating R as a variable depending on T_2, we have

(3) $$\frac{dR}{dT_2} = \frac{2T_1L_1}{A_1E_1} \cdot \frac{dT_1}{dT_2} + \frac{T_2L_2}{A_2E_2}.$$

But, from equation (2),

$$\frac{dT_1}{dT_2} = -\frac{1}{1.414}.$$

Then equation (3) becomes, by substitution,

$$\frac{dR}{dT_2} = -\frac{2T_1L_1}{1.414A_1E_1} + \frac{T_2L_2}{A_2E_2}.$$

Putting this equal to zero and solving for T_2, we get

(4) $$\frac{T_2L_2}{A_2E_2} = \frac{2T_1L_1}{1.414A_1E_1}.$$

But in this case $L_1 = 1.414L_2$. Then equation (4) becomes

$$(5) \qquad \frac{T_2}{A_2E_2} = \frac{2T_1}{A_1E_1}.$$

By use of equations (2) and (5), the values of T_1 and T_2 can now be determined in terms of P.

239. General Remarks. The principles of §§ 237 and 238 furnish the means for the solution of many problems which are statically indeterminate. The reactions of continuous and restrained beams may be determined from the principle of least work and the stresses in trusses with redundant members also can be found. The deflections of framed structures and the reactions of arches are other problems that may be solved by these methods.

While the principles, as here given, are simple, the solutions become quite complex and no adequate treatment of them is possible in a text of this nature. However, it has been worth while to point out the lines along which statically indeterminate problems can be solved, leaving it to the student to carry on the idea with the help of more advanced treatises when necessary. In a number of texts, solutions are worked out to determine the reactions of continuous beams, etc., using the principle of least work.

PROBLEMS

1. A rectangular table has all four legs of the same size and material. The distances on centers of the legs are 6′ 0″ in one direction and 4′ 0″ in the other. A load of 600 lbs. rests on the table, 2′ 0″ from a corner, measured along the length of the table, and 1′ 0″ from the same corner measured along the width. What is the total stress in each leg?

CHAPTER XXIV

MISCELLANEOUS PROBLEMS

240. Stability. A body is said to be stable when it is not easily overturned. The idea is illustrated by Fig. 408, which shows three cones, each illustrating one of the states or classes of equilibrium.* Cone (a) is in ***stable equilibrium.*** Its center of gravity is at the lowest possible point. The application of any horizontal force tending to tip the cone will raise the center of gravity. If such a force is removed, the cone tends to return to its original position.

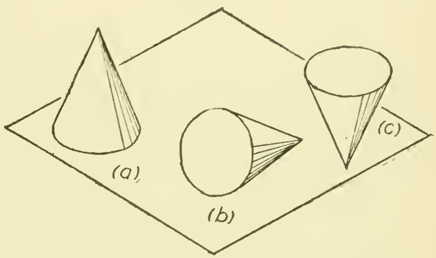


FIG. 408

Cone (c) is in ***unstable equilibrium.*** The center of gravity tends to fall if force is applied.

Cone (b) is in ***indifferent,*** or ***neutral, equilibrium.*** The application of force tends neither to raise nor to lower the center of gravity, and the cone tends neither to return to nor to depart from any given position.

It is needless to say that stability is an important consideration in structural design. The ever-present problem of the overturning moment of the wind is a case in point.

Figure 409A shows a cube resting in stable equilibrium on a plane. The resultant weight G acts through the center of gravity and the resultant reaction R acts through the center of the base. At B, a horizontal force H has been introduced; the center of gravity is rising; R acts through the edge in contact with the

* These three possible *states* of equilibrium should not be confused with the three *conditions* of equilibrium. The latter are conditions necessary to maintain any body in *any* state of equilibrium (§ 32).

plane. The friction F balances H. If H is removed, the moment of G will bring the cube back to its former position. At C, the center of gravity has reached its highest point; G and R

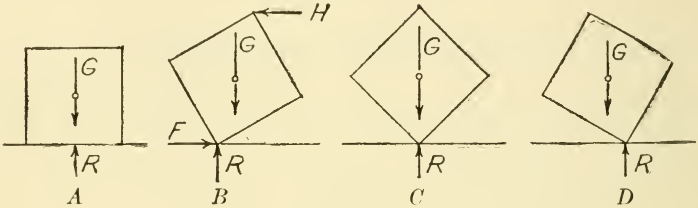


Fig. 409

act in the same line passing through the contact edge and the cube is in unstable equilibrium. At D, G falls to the left of R and the center of gravity is falling, under the action of the negative moment due to G. Unless a force acting toward the right is introduced, it will continue to fall until the center of gravity reaches the lowest possible point and the cube is again in stable equilibrium.

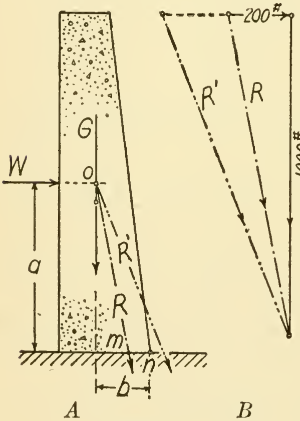


Fig. 410

A typical problem involving stability is that presented by a wall acted upon by wind. This problem may be solved either graphically or analytically, as follows.

A. Graphic Solution. Let Fig. 410 represent the cross section of a wall acted upon by wind which is horizontal and which produces a uniformly distributed pressure on the vertical face. Let it be assumed that the joint beneath the wall cannot carry any tension. Then the weight of the wall is its only source of stability.

Since the overturning force and the weight are both proportional to the length of the wall (perpendicular to the paper), this length is immaterial. Consider a section 1′ 0″ long. Suppose it to weigh 1,000 lbs. and let the resultant force of the wind be 200 lbs.

The line of action of the force of gravity acting on the wall will pass through the center of gravity of the cross section, as shown by G, on the space diagram A. Similarly the force of the wind will be shown by W. The resultant of those forces, when found, will pass through o. The force diagram B now can be constructed and the amount and direction of the resultant of G and W can be determined, as R. Now let a line be drawn on the space diagram, through o and parallel to R in the force diagram. This indicates the line of action of the resultant force acting on the wall. This line cuts the base of the wall at m. The reaction will be a force equal and opposite to R and acting in the same straight line. The *unit* stresses developed on the joint can be determined from § 192B. If the unit stresses are not excessive, the wall is stable.

If W had been twice as great (as shown by dotted lines), the resultant R' would have cut the base outside the foundation, showing that no single reaction furnished by the foundation could balance it, i.e., the wall would fall under the action of such a wind, unless the joint beneath the wall is capable of carrying tension, as outlined below.

B. Analytic Solution. By taking a center of moments at n, the overturning moment of the wind is seen to be Wa, while the moment of stability is Gb. If $Wa < Gb$ and the unit stress (§ 192) is not excessive, the wall is stable without reference to the question of tension on the joint. If the joint cannot take tension, and if $Wa = Gb$, the wall is about to overturn. If $Wa > Gb$, the wall is unstable. If the joint can take tension, the stability of the wall depends on the strength of the joint and the case can be treated as in § 197.

Fig. 411

PROBLEMS

1. In Fig. 411 is given the cross section of a concrete dam 50′ long. The water pressure covers the entire vertical face. Is the dam stable? For unit weight of concrete see Table I of Appendix.
2. A block of granite (175 lbs. per cu. ft.) is 5′ × 5′ at the base and 2′ × 2′ at the top. It is 18′ 0″ high. A rope is attached on the center line of one

face, 9′ 0″ from the ground. The rope leads away at 30° below the horizontal. What pull on the rope is needed to overturn the block?

3. A wall 1′ 8″ thick and 40′ 0″ high is built of common brick masonry (120 lbs. per cu. ft.). What wind pressure (in lbs./sq. ft.) will just overturn it?
4. In Problem 3 above, in what ratio will the stability be increased or decreased if the wall is made 20′ 0″ high?
5. The cross section of a concrete dam is a right triangle 10′ 0″ high and with a base 8′ 0″ wide. The water is retained on the vertical face and is 9′ 0″ deep. (*a*) Is the dam stable? (*b*) If the joint between the dam and the foundation can take tension, what are the maximum and minimum unit stresses on the joint?
6. If the dam in Problem 5 is reversed so that the sloping face receives the water pressure, to what extent are the above results altered?

241. Friction. The study of friction carries more of interest for the mechanical engineer in its applications to power transmission and absorption than it does to the structural engineer, whose problems are chiefly static. This difference is inherent in the subject, since friction always accompanies motion, but when bodies are at rest they are not always, nor even usually, acted upon by frictional forces.

There are, however, certain types of construction, as pile foundations and stone arches, and certain details, such as spikes and rivets, which to a greater or less extent depend upon friction for their proper action. Moreover, even when friction is not considered in the design, it is present in many cases in the completed structure and has its effect on the action of the various parts. Hence some knowledge of the principles involved is important.

Friction is the resistance to motion that exists when two rough surfaces in contact move or tend to move on one another. All surfaces are more or less rough in the sense here intended. Hence no known surfaces are entirely frictionless. In many cases, however, the effect of friction is small *in comparison with* the other forces in the problem, and it may be neglected. Where this is the case, the surfaces are said to be *smooth*.

In Fig. 412 is shown a brick resting on a plank and acted upon by the force P. If P is small, motion will be prevented by the friction F. As P increases, F will increase, up to a certain amount, when sliding will occur. When the body is in motion,

the frictional resistance (measured in pounds) is less than it was just before motion commenced. The friction existing when the body is about to start is called the friction of impending motion (or the friction of rest). Its amount is equal to that of P at the same instant.

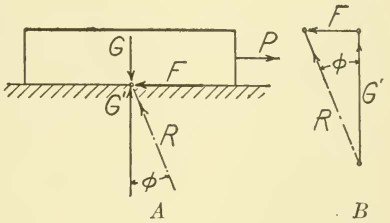


Fig. 412

From what precedes it will be seen that friction is in some sense a reaction, in that before motion starts it is a passive resistance set up by, and equal and opposite to, the active force.

The reactions on the brick in Fig. 412 are then the gravity reaction G' and the friction F. The resultant of these two is R. The angle ϕ will be such that

$$\tan \phi = \frac{F}{G} \cdot$$

Experiments show that the ratio between the friction and the weight of the body (i.e., F/G or $\tan \phi$) is constant for all sizes and weights of bodies, but that it varies greatly with the materials in contact. This ratio is called the *coefficient of friction.* It is usually expressed in the form of a decimal (see Table IV in the appendix). The important thing to remember is that this decimal is really a ratio of friction to weight. The angle ϕ is the angle whose tangent is equal to this decimal.

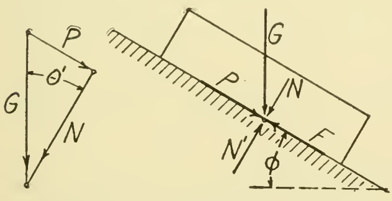


Fig. 413

If a weight is placed on a slanting surface, Fig. 413, the force of gravity G acting on the weight can be resolved into components parallel to and perpendicular to the support, as shown in the drawing by P and N. The reaction of the surface can likewise be resolved into components N' (a normal pressure) and F (friction). If the tangent of the angle θ on the force

diagram is equal to the coefficient of friction for the given surfaces, motion is impending. Therefore, if the surface is inclined to the horizontal at an angle ϕ ($= \theta$) the tangent of which is the coefficient of friction, the body is about to slide. The angle ϕ therefore is sometimes called the *angle of repose;* perhaps "*the limiting angle of repose*" would be better.

From the above, it will be clear that if a body, resting on a rough surface, is acted upon by various forces, and if the *resultant* of those forces makes an angle with a line drawn normal to the surface of contact which is *greater than* ϕ, slipping will occur. If the angle is less than ϕ, no slipping will occur. This principle is of importance in the design of masonry arches, dams, and retaining walls.

PROBLEMS

1. If the block in Problem 2, § 240, rests on a stone base, what is the least horizontal force that will slide the block?
2. If the block in Problem 1 must be pushed along by means of a force acting on the center line of one face and inclined downward at 15° below the horizontal, what is the limiting height below which the application of the force will cause the block to slide and above which it will cause it to overturn?
3. A timber 12″ × 12″ and 16′ 0″ long is placed vertically against a masonry wall and is held in place by the pressure of a horizontal strut applied at the center of the timber. How much pressure from the strut is required to hold the timber in place?
4. How much force is required to start a wooden box 4′ × 4′ and 3′ high and which weighs 500 lbs. down a steel chute which is inclined at 10° to the horizontal?
5. A ladder which weighs 35 lbs. and is 20′ 0″ long rests against a masonry wall and a wood floor. It is inclined at 75° with the horizontal and its center of gravity is 12′ 0″ from the ground. (*a*) Is there enough friction developed to support a man weighing 150 lbs. and who stands $\frac{3}{4}$ of the way up? (*b*) If the wall is smooth? (*c*) If the wall is smooth, what is the least coefficient of friction on the floor that will make equilibrium possible?

242. Non-Coplanar Forces. Most of the problems of structural engineering involve only coplanar forces. But exceptions to this rule are not rare. The following solutions cover some typical cases. The principles involved in these solutions may be used in a large variety of problems. It may be well to note that in general the problem of non-coplanar forces is largely

geometric. The handling of planes and of lines in planes, true angles, etc., requires that the point of view of descriptive geometry be combined with the simple principles of statics, in order to obtain the solutions for these problems.

A. All Forces Concurrent. (1) *Typical case.* A typical case is shown in Fig. 414. Here A is vertical, C is horizontal, and B is inclined to both horizontal and vertical. The resultant can be found by resolving each force into its components parallel to the axes OX, OY, OZ, and adding these components. The force B has a vertical component equal to $B \sin \phi$ and a horizontalc omponent equal to $B \cos \phi$. This horizontal component can be resolved into components parallel to OZ (that is, $B \cos \phi \cos \alpha$), and to OX (that is, $B \cos \phi \sin \alpha$). It follows that the components of the resultant will be $A + B \sin \phi$, in a direction parallel to OY; $C \sin \theta - B \cos \phi \sin \alpha$, in a direction parallel to OX; and $C \cos \theta + B \cos \phi \cos \alpha$, in a direction parallel to OZ.

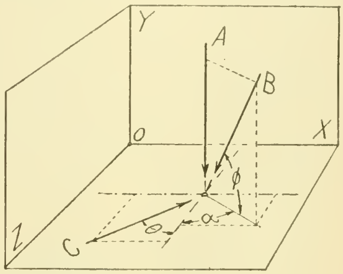


Fig. 414

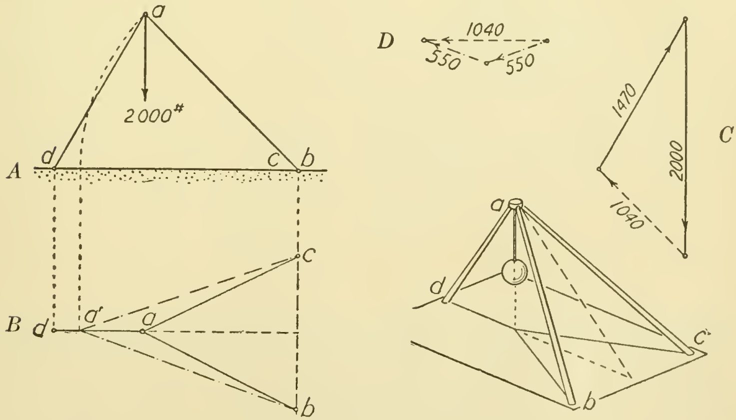
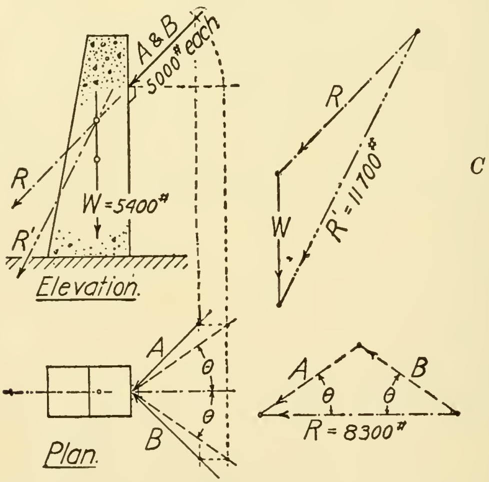


Fig. 415

A problem of this type may involve three or more forces, and the solution may be made either analytically or graphically.

(2) *A tripod.* As another case let it be required to find the stresses in the tripod shown in Fig. 415. Draw the view A and consider that the load is supported by the leg ad, and a resultant force in the plane of the legs ab and ac. From symmetry, we know that this resultant will be concurrent and coplanar with the load and the leg ad. Draw a force diagram C for these three forces. The stress in ad is found to be 1,470 lbs. and the resultant stress in the plane abc is found to be 1,040 lbs. Revolve the triangle abc into the horizontal, as shown on view B. This gives the true angle between the legs ab and ac. Now construct force diagram D, starting with the force of 1,040 lbs., parallel to the median line of $ba'c$. The stresses in ab and ac are thus found to be 550 lbs. each.

(3) *A pier.* The pier shown in Fig. 416 is another case. Let it be required to investigate the stability (§ 240) of the pier under



A Fig. 416 B

the action of the forces A and B and of its own weight, which is expressed by W. The forces A and B are concurrent and coplanar, the angle between them being 2θ. Their resultant is 8,300 lbs., as shown by R. This resultant is concurrent and coplanar with the resultant weight of the pier. The final resultant R' passes outside the base, as shown.

B. Some of the Forces Non-Concurrent. 1. *An analytic solution.* In Fig. 417 is shown a door acted upon by a force of 156 lbs. inclined to all three of the planes of projection. Let it be required to find the amount and direction of each of the reactions. This is the same door that was used in Problem 5, § 40, but in that case the forces were coplanar. It will be noted that, because of the way the hinges are constructed, the top hinge cannot offer any vertical reaction. At d there is a frictionless bumper. This can exert only a force perpendicular to the plane of the door.

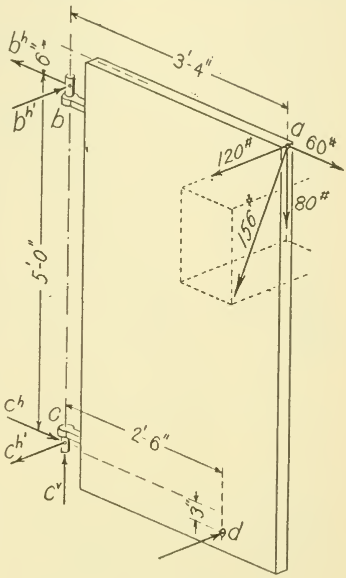


Fig. 417

In Fig. 417, the door is shown in isometric projection with the 156 lb. force resolved into its three components. The reactions at b, c, and d are also shown by their components. The arrows on these component reactions are determined by inspection. If a mistake should be made in indicating these arrows, the later computations would give a negative value for the amount of the reaction, which would indicate that the assumed condition should be reversed. Now applying the conditions of equilibrium, we find

$$\Sigma V = 0 \text{ gives} \quad c^v = 80,$$
$$\Sigma H = 0 \text{ gives} \quad b^h = c^h + 60,$$
$$\Sigma H' = 0 \text{ gives} \quad d - c^{h'} + b^{h'} = 120.$$

Taking moments in the plane of the door and about the point c, we find

$$5b^h = 80(3\tfrac{1}{3}) + 60(5\tfrac{1}{2});$$

taking moments about bc

$$2\tfrac{1}{2}d = 3\tfrac{1}{3}(120);$$

and lastly, taking moments perpendicular to the plane of the door and about the point c,

$$\tfrac{1}{4}d + 120(5\tfrac{1}{2}) = 5b^{h'}.$$

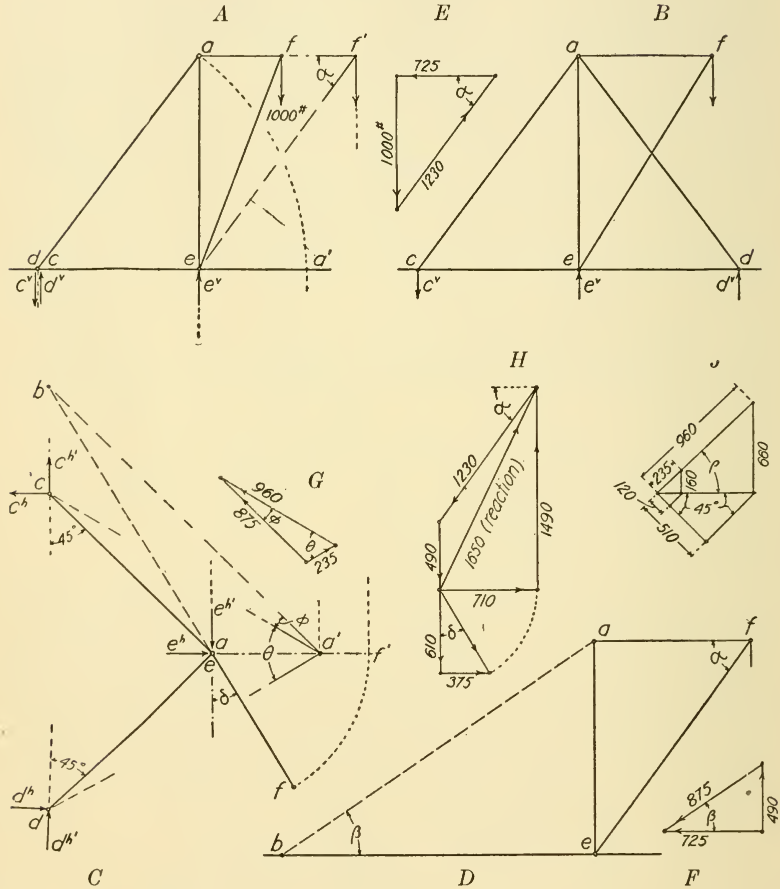


Fig. 418

From these six equations, the values of the six unknown quantities are $b^h = 119\frac{1}{3}$, $b^{h'} = 140$, $c^h = 59\frac{1}{3}$, $c^{h'} = 180$, $c^v = 80$, $d = 160$, all with the directions indicated by the arrows in Fig. 417.

2. *A graphic solution.* Given a stiff-leg derrick, as shown in Fig. 418. Let it be required to determine the stresses in the

members and the reactions, in terms of their components. Drawings *A*, *B*, *C*, and *D* are space diagrams. The others are force diagrams.

In view *A* let the boom be swung around parallel to the *V* plane. This gives the angle α as the true angle between the boom *fe* and the top stay *fa*. By the ordinary solution for concurrent forces (§ 25), force diagram *E* is constructed, giving the stresses in *fe* (1,230 lbs., compression) and *fa* (725 lbs., tension).

Next let a vertical plane be passed through *fa*, *fe*, and *ea*. It will cut the plane of the stiff legs in the dotted line *ab*. A projection of these members in their own plane is given in drawing *D*. The stress in *af* and *ae* could be equilibrated by a force in the line *ab*. By taking a free body about *a* in drawing *D*, we can draw the stress diagram *F*, which shows the stress in the mast *ae* (490 lbs., compression) and the necessary equilibrant in the line *ab* (875 lbs., tension).

In the actual derrick, the stiff legs (*ac* and *ad*) furnish the equivalent of the force *ab*, just determined. Let the lines *ab*, *ac*, and *ad* be rotated down into the *H* plane (on drawing *C*). The point *a* falls at a'. The real angles between these lines (θ and ϕ) are shown at a'. The force determined above as acting in the line *ab*, and the stresses in the stiff legs, give rise to the force diagram *G*, which shows the stress in *ac* (960 lbs., tension) and in *ad* (235 lbs., compression).

Drawing *H* is a force diagram in which the stresses in the boom and mast are combined to get the reaction at *e*, and the reaction is split into its components. Drawing *J* is used to resolve the reactions due to the stiff legs into their components. The angle ρ is the true angle *ac* and *ad* make with the horizontal plane.

Such methods as these find frequent application in the design of the piers, buttresses, etc., in connection with Gothic trussing and vaulting.

PROBLEMS

1. Is the pier in Fig. 419 stable?
2. Find the stresses in the members *ab, ac, ad, af, fe* in the derrick in Fig. 420.
3. Find the stresses in the tripod, Fig. 421.
4. Let the boom of the derrick in Problem 2 be swung through a horizontal angle of 30°. Find the stresses in the members.

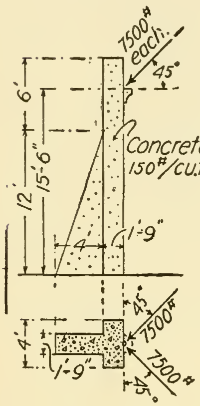


Fig. 419

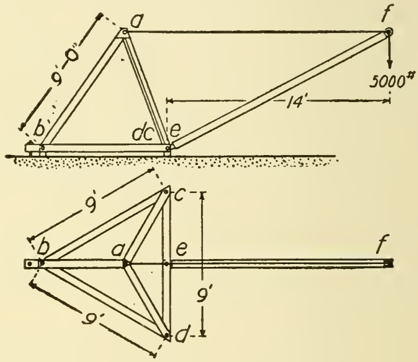


Fig. 420

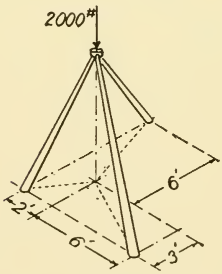


Fig. 421

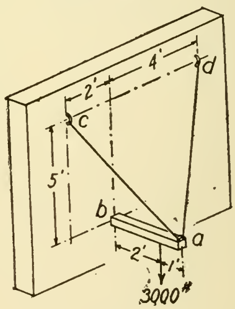


Fig. 422

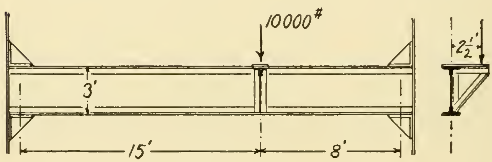
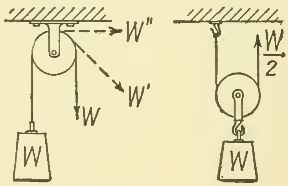


Fig. 423

5. Find the stresses in the members *ab, ac, ad* of the frame in Fig. 422.
6. In Fig. 423, determine at least one set of forces, external to the beam, which will maintain equilibrium.

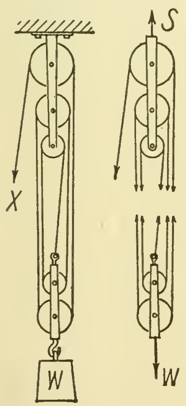
7. In Fig. 417, let the same force be applied at the *center* of the door. Find the component reactions.
8. Let the boom of the derrick in Fig. 418 be swung around till its plane makes an angle of 30° with aa'. Determine the stresses and component reactions.

243. Pulleys. Everyone is familiar with the fact that by the use of pulleys a relatively small force may be made to lift a large load. The force required in any given case as well as the stresses in the ropes can be determined by the free body method (§ 23). In Fig. 424*A*, which represents a single pulley, it is evident that the load W can be held in equilibrium only by a force equal to W, since each force acts at the same distance from the axle. Moreover, this equality must hold for any direction taken by the supporting force, as W, W', W''.



A FIG. 424 *B*

In the case of a moving pulley, Fig. 424*B*, the weight is supported by two ropes, the tension in each being $W/2$.



A *B*
FIG. 425

In Fig. 425*A*, the load W is supported by the force X. Let the amount of X be required. In Fig. 425*B* the two sets of blocks are shown as free bodies, and it is at once evident that all the stresses in the rope must be equal and each equal to $W/5$. The pull on the support S will be $6W/5$.

Another way to approach a problem concerning pulleys is to consider the total work (§ 234) done by the force and the energy stored in the 'oad. Thus in Fig. 424*B*, the force $W/2$, in moving through some distance x, performs work amounting to $Wx/2$ ft. lbs. Meanwhile the load rises through $x/2$ ft. and the potential energy stored in it is $Wx/2$ ft. lbs.

PROBLEMS

1. What force P is needed to support the weight in Fig. 426?

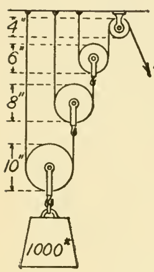


FIG. 426

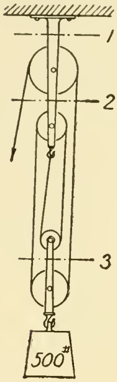


FIG. 427

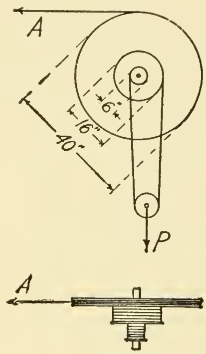


FIG. 428

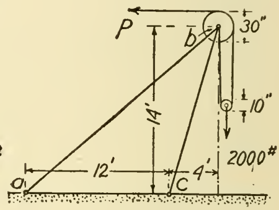


FIG. 429

2. What are the stresses in the pulley frames in Fig. 427 at each of the sections indicated as *1*, *2*, and *3* ?
3. Figure 428 shows a differential pulley. If the rope A is pulled by a force of 25 lbs. acting through 12′ 0″: (a) What is the greatest load P that can be lifted; (b) How far will it rise?
4. In Fig. 429, the force P is just enough to lift the load. What are the stresses in ab and bc ?

244. Temperature Stresses. Nearly all materials expand with a rise in temperature and contract as they cool off.* Careful investigation has shown that the amount of this change in size is proportional to the change in temperature, and also to the size of the piece. It takes place equally in all directions when the material is homogeneous.

For each material there is a number, called the ***coefficient of linear expansion,*** which expresses the change in length (or breadth or thickness) per degree of change in temperature, per unit of length (or breadth or thickness). These coefficients are quite small decimals. Some reference books quote the coefficient as for 100° change in temperature. Every table of coefficients should therefore state clearly whether it is made up for the

* The exception of water between 39° and 32° F. is sufficiently important to deserve notice.

Centigrade or the Fahrenheit scale of temperature, and for how many degrees of change according to that scale. (See Table I, Appendix.) Coefficients of surface expansion and of volumetric expansion are sometimes quoted, but are not important for the work in hand. If needed they can be derived from the coefficient of linear expansion.

If any piece of material is homogeneous and free to move in all directions, change in size due to temperature is accomplished without resultant stress. But when the material is restrained so that its normal adjustment to temperature change is prevented, stresses are set up within the piece. The nature and amount of these stresses will depend on how the restraint is imposed and on whether the restraint is complete or partial. The following cases are the more important ones.*

A. Homogeneous Material, Restrained. In Fig. 430, let the bar *ab* be solidly fastened to the two rigid piers *A* and *B*. Suppose that the temperature falls and that the bar *tends* to shorten to the length *ac*. The shortening is prevented by the rigidity of the connections, and the length remains *ab*. The stress thus set up in the bar is the same as *would have been* caused by elongating the bar from the length *ac* to the length *ab*, by means of a direct pull.

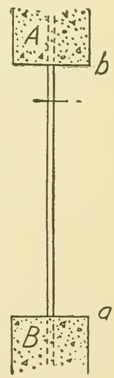


Fig. 430

In order to illustrate the principle, let us suppose that Fig. 430 represents a bar of medium steel (coefficient of expansion per degree Fahrenheit, 0.0000067) and that it is 20 feet long and 1 inch in diameter. Suppose it to be fastened between the rigid piers *A* and *B* when the temperature is 70° F. and that later the temperature falls to 0° F. What, at that time, is the stress in the bar?

The normal shortening of the bar (*bc*, Fig. 430) is

$$70 \times 0.0000067 \times (20 \times 12) = 0.1126''.$$

* The following discussion uses the principle of proportionality of stress and deformation, and hence does not apply to stresses beyond the elastic limit. See also footnote on page 396.

The modulus of elasticity for this material is 29,000,000 lbs. per sq. in. In order to produce the above change in length, the unit stress in the bar must be equal to x in the following equation (§ 70):

$$29{,}000{,}000 = \frac{\dfrac{P}{0.7854}}{\dfrac{0.1126}{240}} = \frac{x}{\dfrac{0.1126}{240}}, \qquad x = 13{,}600 \text{ lbs. per sq. in.}$$

The *total* pull exerted by the bar is $13{,}600 \times 0.7854 = 10{,}700$ lbs.

In order to generalize what precedes, let us put the quantities down in symbols. Let L be the length of the bar, c the coefficient of expansion, and t the change in temperature. Then the normal change in length $= ctL$, and

$$E = \frac{\dfrac{P}{A}}{\dfrac{ctL}{L}} \qquad \text{or} \qquad \frac{P}{A} = Ect.$$

From this expression, it is clear that the *unit* stress set up in a restrained bar depends on the material and the change in temperature only, and is independent of the length and cross-sectional area. Stresses due to rising temperature are computed in the same manner, but are compressive instead of tensile.

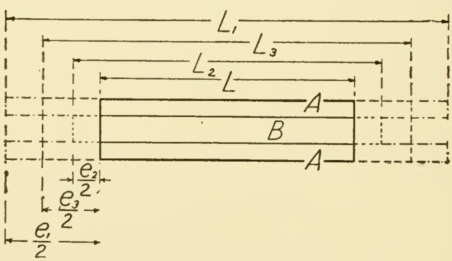


Fig. 431

B. Non-Homogeneous Material, Unrestrained. Let Fig. 431 represent a bar of length L, made up of two bars A having a high coefficient of expansion and one bar B having a low coefficient. The bars A are alike as to size and material and the

following solution depends on this symmetry. Let the bars be rigidly fastened along the adjoining faces and let the whole piece be unrestrained. Now let the temperature rise.

The bars A, if alone, would elongate to some length L_1; the bar B to some length L_2; and the three being fastened together will actually elongate to some length L_3, intermediate between L_1 and L_2. When in this latter condition, the tendency of the bar A to become even longer will have stretched the bar B through a distance $(e_3 - e_2)$, producing in it tensile stresses corresponding to the actual abnormal elongation. Similarly the tendency of B to lag behind A produces in A compressive stresses corresponding to the shortening $(e_1 - e_3)$. Manifestly the total tension in B and the total compression in A are in the nature of action and reaction and they are therefore equal.

Let it be required to find the actual final length L_3 and the unit stress in each bar. Let c_1 and c_2 be the coefficients of expansion of A and B; let A_1 and A_2 be the cross-sectional areas of A and B; let E_1 and E_2 be the moduli of elasticity of A and B, and let t be the rise in temperature.

(1) The natural elongation of A will be:* $e_1 = c_1tL.$

(2) The natural elongation of B will be: $e_2 = c_2tL.$

The total compressive force needed to shorten A through the distance $(e_1 - e_3)$ will be

$$P_1 = \frac{E_1A_1(e_1 - e_3)}{L}, \tag{3}$$

and the total tensile force needed to stretch B through the distance $(e_3 - e_2)$ will be

$$P_2 = \frac{E_2A_2(e_3 - e_2)}{L}. \tag{4}$$

Equating these two values and solving for e_3, the actual elongation of the combination, we get

* In these equations the *unit* deformations are figured as if L, L_1, L_2, and L_3 were equal. The error thus introduced is negligible when L is large in comparison with the change in length.

$$(5)\qquad e_3 = \frac{E_1A_1e_1 + E_2A_2e_2}{E_1A_1 + E_2A_2}.$$

We now have five equations, which, taken together, can be made to yield the unit stress in each material and the final length for any change in temperature.

The student should not accept these derivations without following them through in detail for himself. It is well to note that here again the unit stresses are independent of the length; also that a fall in temperature would call for certain changes in signs in the equation. Further, it should be noted that the above action cannot take place unless the bars A are fastened to the bar B in a manner capable of developing a shearing resistance equal to the total tensile or compressive forces, as given in equations (3) or (4).

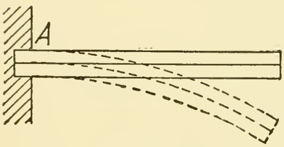


Fig. 432

If only two bars are fastened together, the tensile and compressive forces are eccentric and cause a deformation like that shown in Fig. 432. This principle is largely used in thermostats. One end of the bar being held fast, as at A, Fig. 432, the other end is moved by temperature change and this motion is used to control an electric circuit.

C. Non-Homogeneous Material, Restrained. This case can be studied in two parts: (1) when no fracture occurs, and (2) when one material is fractured, the other remaining intact.

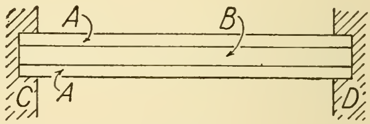


Fig. 433

(1) *No fracture.* In this case the unit stresses are the same as they would be in separate homogeneous bars, and no stress is brought on the bond between the bars. Let the student prove this statement, remembering that the end fastenings are assumed to be *fixed* in position.

(2) *One part fractured.** In Fig. 433, let the bars A and B

* In the following discussion, the proportionality between stress and deformation is assumed to extend to failure. But since no quantitative results are made

have the same coefficient of expansion, but let A have much lower strength and modulus of elasticity in tension than has B. Let the bars be fastened to the rigid piers C and D and let the temperature fall. Then from A and C (1) above, the stress in each bar can be determined for each fall of temperature, until the stress in A reaches the ultimate strength of the material. When that happens, the bars A will crack (at the weakest section) as at a, Fig. 434. The subsequent developments will depend on the adhesion or bond between the bars. If this adhesion is weak and gives way, the entire normal contraction in bars A may take place at once (as in Fig. 435), leaving bar B carrying the stress appropriate to the actual temperature and bars A

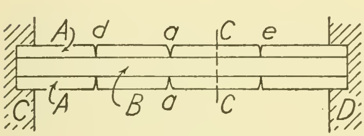


Fig. 434

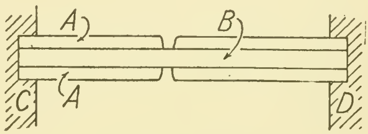


Fig. 435

fully contracted and unstressed. If the bond is strong and resists the motion natural to bars A (as in Fig. 434), the crack will widen toward the top (which is unrestrained) just enough to relieve the stresses in the separate parts of the bars A to a point below the ultimate strength. This will leave each half of bars A stressed nearly to the breaking point (since the unit stress does not depend on length) and the bond between A and B will be called upon to resist a total force nearly equal to the ultimate strength of the bars A. We can get an approximate idea of the stress cond tion by taking as a free body that part of the bar which lies between sections aa and CC as shown in Fig. 436. The arrow F indicates the temperature stress in B, while F' indicates the stress in each of the bars A. We have shown just above that F' is about equal to the ultimate strength of bar A. Therefore F'', the stress

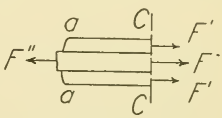


Fig. 436

to depend on this assumption, the principles deduced are not seriously affected thereby.

in B at the section where A is ruptured, will be equal to $F+2F'$. Moreover, the bond between bars A and B must be offering a shearing resistance equal to F', on each of the surfaces of contact.

Now, turning to Fig. 434, if the temperature continues to fall, a new crack may be opened up at some new section, say at e or d, and the stress condition between the cracks will again be as in Fig. 436.

Although the preceding analysis is not quantitative in any sense, it is sufficient to show that in such a case (1) the total stress in bar B is roughly equal to its own normal temperature stress plus the ultimate strength of bars A; (2) if the bond between the bars is strong, the normal shrinkage of bars A will cause many small cracks rather than one large one.

The design of *temperature reinforcement for concrete* is based on the principles developed above. Because of the defective elastic properties of concrete and the fact that the stresses involved are normally those producing rupture, perhaps no satisfactory rational method for computing the amount of steel needed to prevent unsightly cracking is possible. Certainly none is in common use. The general principles underlying the empirical rules in ordinary use are as follows:

(1) If a piece of reinforced concrete is free to expand and contract (unrestrained), no cracking will occur, since the coefficients of expansion of the two materials are nearly equal.

(2) If the piece is *completely* restrained, the concrete will crack at a very small drop in temperature (10° to 20° F.). This is because of the very low tensile strength of concrete. When such cracking occurs, the stress in the steel is relatively small and no amount of reinforcement can prevent the cracks from forming in the concrete.

(3) However, because of its bond (§ 90), the steel can prevent the cracking from taking place at one spot, as in Fig. 435, and force it to occur at many places, as in Fig. 434. Thus each crack will remain small and unnoticed. Since a large bond strength is desirable, the steel used is usually in the form of many small rods rather than a few large ones.

(4) The nearer the surface the steel is placed the less is the tendency for the crack to spread toward the surface, as at *a*, Fig. 434.

(5) The steel used should be sufficient in amount so that it will not be stressed above its elastic limit, thus permitting individual cracks to enlarge. A computation of the amount needed to produce this result can be made as follows: Let the ultimate strength of the concrete in tension be *200 lbs. per sq. in., and let the area of cross section of concrete in our piece be A_c. Then the stress required to rupture the concrete will be $200A_c$. Let the modulus of elasticity of the concrete be *1/15 of that for steel. Then from § 207, the unit stress in the steel just before the concrete cracks will be $*15 \times 200 = 3{,}000$ lbs. per sq. in., and if the area of the steel is A_s, the total stress in the steel is $*3{,}000A_s$. After the concrete cracks, we wish to have the stress in the steel less than the elastic limit, say 36,000 lbs. per sq. in. Then (see Fig. 436)

$$200A_c + 3{,}000A_s = 36{,}000A_s,$$

whence

$$\frac{A_s}{A_c} = 0.006.$$

This would seem to indicate that the area of steel should be about 0.6 per cent. of that of the concrete, but as already pointed out on page 398 and in the footnote on page 396, the result of any such computation is open to serious question. In practice it is found that from 0.2 per cent. to 0.4 per cent. of steel will suffice, but of course the conditions are seldom those of absolute restraint.

While the above computation leads to no important result, the form of the equation shows plainly that there is a definite advantage in using for this purpose a steel having a high elastic limit.

PROBLEMS

1. A steel bar 2″ in diameter and 25′ 0″ long is placed between two rigid connections at a temperature of 70° and without initial stress. What will be the unit stress in the bar when the temperature falls to 0°?

* None of these quantities can be stated except as an approximation.

2. In Problem 1, if the bar is $1\frac{1}{2}''$ in diameter, what will be the reaction at each connection?
3. A steel rod 20′ 0″ long connects two rigid supports. It is put in place at a temperature of 70° and at that temperature it carries a tensile stress of 50,000 lbs. The temperature may fall to zero degrees. What must be the diameter of the rod?
4. A steel strut 16′ 0″ long is put in place between two rigid piers at a temperature of 60° and at that temperature it must carry a total compression of 40,000 lbs. The temperature may rise to 90° or fall to − 20°. Design the strut and compute the strength needed at the connections.
5. A bar of aluminum 1″ × 4″ is rigidly fastened between two bars of steel each $\frac{1}{2}''$ × 4″ at a temperature of 50°. What is the unit stress in each material when the temperature rises to 100°?
6. A rod of steel 1″ in diameter and 8′ 0″ long has a sheathing of brass $\frac{1}{4}''$ thick which is put in place at a temperature of 300°. What is the stress in each material when the temperature is 50°?

245. Cylinders. The methods for investigating the stresses in cylinders vary with the thickness of the cylindrical walls. We will not here consider the case of thick cylinders, such as gun barrels. For the thin-walled cylinder, such as is usual in the case of pipes, water tanks, etc., the following principles will suffice:

Let a pipe be subjected to internal pressure from a fluid or gas (water or steam). From the principles of physics the pressure is known to be of equal intensity in every direction, as shown in Fig. 437*A*. Let the cylinder be cut by a vertical plane and one half of it be shown as a free body in Fig. 437*B*. The material (liquid or gas) which fills the half pipe is under a pressure from that in the other half the intensity of which is the same as that throughout the entire mass. This situation is indicated by the horizontal arrows on the diameter of the section. Evidently the tensile stresses in the cylinder walls, indicated by the arrows marked "*S*," must be equal to the pressure of the fluid on the diametral plane.

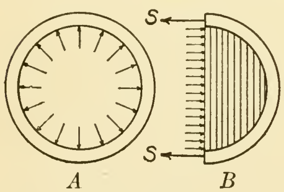


Fig. 437

If the pressure on the cylinder is exterior, rather than interior, the stresses are compressive instead of tensile. However, these

compressive stresses act on the cylinder wall somewhat as in the case of a column. They tend to deform the walls unsymmetrically and to collapse the cylinder at stresses less than the crushing strength of the material.

PROBLEMS

1. What unit stress is set up in a steel pipe 4″ in diameter and 3/32″ thick by a steam pressure of 100 lbs. per sq. in.?
2. A water tank 8′ 0″ in diameter and 10′ 0″ high is made of wooden staves and steel bands. The bands are of 5/8″ diameter, threaded. Determine the proper spacing intervals for the rods, as determined by the water pressure.
3. A pipe line 4′ 0″ in diameter is laid on a 1 per cent. grade through a distance of one half a mile. It is made of wooden staves and steel bands of 5/8″ diameter, threaded. What is the necessary spacing for the bands at each end of the line, assuming that their sole function is to retain the water pressure?

246. Torsion. When a moment acts in a plane normal to the longitudinal axis of a piece of material, it is called a *torsional* (or twisting) moment, and the resulting stress is called torsional stress. The study of the phenomena of torsion is of prime importance in connection with the transmission of power through shafting. The structural engineer has few problems of importance which involve torsional stresses, but some knowledge of the general principles involved is desirable. We will here discuss only one case, viz., that of a shaft of circular cross section.

Figure 438 shows a shaft solidly fixed at one end and acted upon by a moment Pa, at the other end. The line oc is a radius of the shaft and cd is an element of the surface. Both these lines are drawn before twisting occurs. As the forces are applied, the moment arm rotates; oc takes the position ob and the straight line dc becomes the helix db. Evidently shearing stresses are set up on any plane passed normal to the longitudinal axis of the shaft.

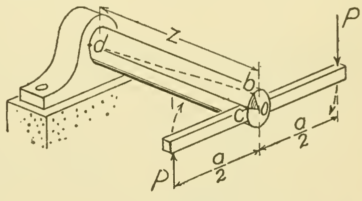


FIG. 438

Tests bring out a number of important facts which are anal-

ogous to those having to do with the performance of materials under tensile and bending stress.

(1) The *angle of torsion, boc,* is proportional to the twisting moment up to a definite elastic limit. When this limit is exceeded, the deformation increases faster than the applied moment and the phenomenon of a permanent set appears just as in tensile tests (compare § 66). Therefore a stress-deformation curve, similar to Fig. 146, may be constructed to express the results of a torsional test.

(2) For round bars a straight radial line like *oc*, Fig. 438, is found to remain straight throughout the test, provided the elastic limit of the material is not exceeded. (Compare the maintenance of plane sections in beams, § 130.)

(3) These facts, coupled with the necessary conditions of equilibrium, make it clear that the shearing stresses on any cross section set up a *resisting moment* which must be equal to the *twisting moment.* (Compare equation (2), § 133.)

(4) The twisting moment is the same on every cross section of the shaft and hence the unit shearing stresses are the same for corresponding points on all sections.

(5) The amount of the unit shearing stress at any given point on a cross section is proportional to the distance of that point from the center of the section.

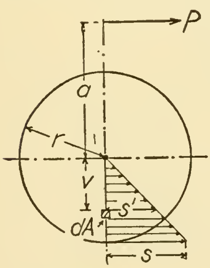


Fig. 439

(6) The direction of the shearing stresses at any such point is perpendicular to a radius drawn through that point.

It should be particularly noted that these conclusions are drawn only for the case of a circular shaft acting under stresses which do not exceed the elastic limit.

Figure 439 shows the cross section of a circular shaft under a twisting moment *Pa*.

The resisting stresses acting on a given radial line are shown as varying from *s* to zero. Let *dA* represent any elementary area distant *v* from the center of the shaft and let *s′* be the unit stress on that element. Then from similar triangles

$$\frac{s}{s'} = \frac{r}{v}, \qquad \text{or} \qquad s' = s\frac{v}{r};$$

and the total stress on the elementary area is $(sv/r)dA$. The resisting moment due to this total stress is then $(sv^2/r)dA$ and the total resisting moment of the shaft is $\int_A (sv^2/r)dA$. But since s and r are constants, this expression can be written $(s/r)\int_A v^2 dA$. This gives the relation

(1) Twisting moment = Resisting moment $= \frac{s}{r}\int_A v^2 dA.$

The quantity $\int_A v^2 dA$ which occurs in this expression is known as the *polar moment of inertia*, it being the same as the rectangular moment of inertia heretofore used except that the distance term is measured from a point instead of a line. If we let I_p represent polar moment of inertia, equation (1) becomes

(2) $$\left.\begin{array}{c}\text{Twisting moment}\\ \text{or}\\ \text{Resisting moment}\end{array}\right\} = M = s\frac{I_p}{r}.^*$$

The polar moment of inertia for a circle is given by† $I_p = \pi d^4/32$. When this substitution is made, equation (2) gives a means for the discussion of stresses in circular shafting.

PROBLEMS

1. What is the resisting moment of a steel shaft 4″ in diameter (*a*) as against torsion, (*b*) as against bending?
2. A steel shaft 4″ in diameter has a wheel 3′ 0″ in diameter mounted on it. What unit stress will be set up in the shaft by a force of 500 lbs. acting tangent to the rim of the wheel?

* Compare the above derivation with that for stress in a beam, § 134.

† The derivation of this value is as follows (see Fig. 440):

$$I_p = \int v^2 dA \qquad \text{but} \qquad v^2 = x^2 + y^2.$$

Therefore

$$I_p = \int x^2 dA + \int y^2 dA.$$

That is, in general, the polar moment of inertia referred to a given point is equal to the sum of the moments of inertia taken about two rectangular axes which intersect at the given point. Thus for a circle $I = \pi d^4/64$ and $I_p = 2 \times (\pi d^4/64) = \pi d^4/32$.

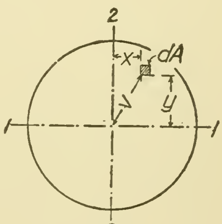


FIG. 440

CHAPTER XXV

SPECIAL GRAPHIC METHODS

247. Introduction. In the preceding chapters the principles underlying graphic methods have been developed and the methods themselves have been used to some extent (§§ 9, 12, 14, 18, 27, etc.). In the simple cases thus far treated the principles can be applied with little difficulty, but as the conditions of the problem become more complex, it becomes more important to study the method *as a method,* and to develop a technique that will simplify the necessary processes.

This chapter is intended to give the student some familiarity with the essentials of this technique. Any real *proficiency* in graphics will involve the study of many more cases than can be presented in this book. But the cases which have been included cover quite well the absolutely necessary foundation for further study. In more advanced texts, methods are developed for handling problems of an intricate nature by means of graphics. Problems dealing with center of gravity, moment of inertia, bending moments, deflections, eccentric loading, and many other complex conditions can be readily solved. To the student whose mind is adjusted to graphic expression, this field is intensely interesting, and the results obtained are most valuable.

248. General Principles. Graphic solutions of problems in statics depend on the fact that a line has the same three characteristics as a force (amount, direction, and position) and that therefore a line can fully express a force (§ 12). In the study of concurrent and non-concurrent forces, we have already seen how these three characteristics give rise to two kinds of diagrams, amount and direction being shown on a ***force diagram,*** while direction and position are shown on a ***space diagram*** (§ 26).

The force diagram necessarily will be made to some scale of pounds per inch, while the space diagram will be at a scale of

inches (or feet) per inch. But the *direction* of each force appears on each diagram. Thus the two diagrams become interdependent and of almost no value if considered separately.*

249. Advantages and Limitations. Graphic methods have several outstanding advantages for certain classes of work. When the problem contains many quantities, and particularly when the amounts of the forces and distances or of the angles involved are not expressible in "round numbers," graphic methods effect important economies. In fact some problems that would be almost hopelessly intricate and tedious in an analytic solution are readily and quickly solved by graphics.

In many cases the results of a graphic computation are self-checking. That is to say, if an incorrect process is introduced, the subsequent processes cannot be carried out. Moreover, the eye often can detect, through the sense of proportion, errors that might be entirely overlooked in an analytic computation. These facts tend to give a person greater confidence and ease in his work; these are important factors in solving long and intricate problems.

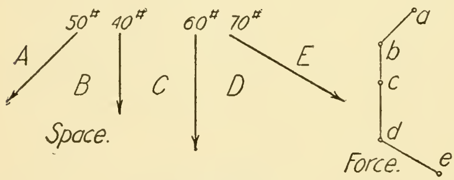
From the point of view of mental training, there is nothing that can so develop a *sense* for structure and assist in the real comprehension of the relationship between the parts of a structure as a graphic expression which appeals at once to the mind and to the eye.

A graphic computation cannot be made with the same degree of precision as a computation in figures. But it is usually possible to attain a greater degree of accuracy than exists in the data of a structural problem (§ 4). This matter of accuracy is, of course, bound up with the question of scales to be used. The eye cannot distinguish easily any division on a scale smaller than 1/50″ to 1/100″. This fact alone introduces a probable error of about

* The fact that the direction characteristic is *common to the two diagrams* frequently leads beginners to attempt to consolidate a force diagram and a space diagram into one. While this can be done in certain cases, it is seldom wise to do so. Space relations are different in kind from force relations and the two cannot be made commensurate. Therefore any attempt to make one diagram cover both space and force relations involves a double scale arrangement which is usually confusing and seldom economical of time or effort.

$\frac{1}{4}$ per cent. into the scaled length of a line 6″ long. Other factors of error are the widths of lines, relative accuracy of instruments, and all those details which make up draughtsmanship. On the whole it is not difficult to keep the probable error well below 1 per cent. The greater the actual dimensions or forces concerned, the smaller may be the scale used, and vice versa. One point that is frequently overlooked is the need of relationship in scale between the space diagrams and the force diagrams. In general, that drawing (whether it be the space diagram or the force diagram) which contains the shortest lines, on the average, will determine the degree of accuracy of the results. This of course means that accuracy cannot be increased by increasing the scale of one diagram only.

In certain problems, the relations involved are so simple and direct that there is no advantage in the graphic method. It takes some time to assemble the apparatus and clear it away. Meanwhile it may be possible to complete an analytic solution. In other problems, the best results are to be had from solutions that are partly analytic and partly graphic. A person must have had considerable experience with both methods before he can wisely choose between them. Neither method should be relied on to the exclusion of the other.



A Fig. 441 *B*

250. Bow's Notation. In the study of various branches of geometry, the student has already encountered the necessity for a careful and consistent system of notation in graphic problems. The following system is not urged as the only possible one but it has proved its value in use and it will be used throughout this chapter. In Fig. 441 we have four forces shown in position on the space diagram and in amount on the force diagram. On the space diagram the force at the left which occurs *between* the letters A and B is known as the force AB. The same force is

shown at the top of the force diagram. It is lettered at its *ends*, using the same letters as before but employing the lower case letters *ab*. Thus, whether we speak of the force *AB* on the space diagram or *ab* on the force diagram, the line representing the force occurs *between* the letters *AB* or *ab*. This system has many advantages which cannot now be made apparent but which will clearly appear as the applications are developed.

251. Resultant of Two Forces.* A. CONCURRENT FORCES, SPECIAL CASE. The general principles for the solution of problems in concurrent forces are given in Chapter III and need not be re-stated here. But it may be worth while to introduce a problem which involves a special method for the solution of cases in which the point of concurrence is inaccessible.

In Fig. 442, let the forces *AB* and *BC* be given, as shown on the space and force diagrams, and let it be required to find the amount and position of their resultant, due regard being paid to the limits of the drawing as shown. The *amount and direction* of the resultant can be found on the force diagram as shown by *ac*. The *position* of the resultant could easily be determined on the space diagram if the forces *AB* and *BC* could be continued to the point *k*. In that event a line drawn through *k* parallel to *ac* (in the force diagram), as *km*, would give the position of the resultant of *AB* and *BC*. The problem then is to *locate* the resultant force without the use of the point *k*.

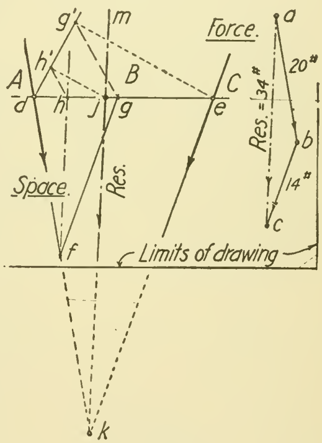


FIG. 442

Choose some point *f* which is on *AB* and within the limits of the drawing. Let *fg* be drawn parallel to *BC*, let *fh* be drawn parallel to *ac*, and let *de* be drawn anywhere so that it cuts the lines *AB*, *BC*, *fg*, and *fh*. We have now created a triangle *dfg*, within the

* For the more general case see § 255.

limits of our drawing, which is similar to *dke* and which contains a line (*fh*) which is placed similarly to the line of the resultant force (*km*) in the larger triangle. From another point of view we may say that we have moved the force *BC* closer to *AB*, thus in effect merely reducing the scale of the space diagram. Now in this new (reduced) diagram the point *f* represents *k* which formerly was inaccessible.

A consideration of the similar triangles *dfg* and *dke* shows that if we can divide *de* into two parts by some (unknown) point *j*, such that $dj/de = dh/dg$, then the position of the resultant of *AB* and *BC* will be known. To do this, draw dg' equal to *dg* and in any convenient direction. Lay off dh' equal to *dh* and connect g' with *e*. Next draw $h'j$ parallel to $g'e$. Now from similar triangles, *dj* is to *de* as dh' (or *dh*) is to dg' (or *dg*). This shows that *j* is in the desired position. The required resultant passes through *j* and is parallel to *ac*.

B. Parallel Forces. In Fig. 443, let *AB* and *BC* be given in position, amount, and direction as shown, and let the position and amount of the resultant be required. Draw *de*, connecting any two points on *AB* and *BC*. From § 41 we know that the resultant will cut this line at some point *h*, such that $dh/he = 2\frac{1}{2}/5\frac{1}{2}$.

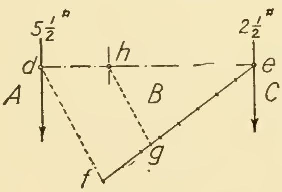


Fig. 443

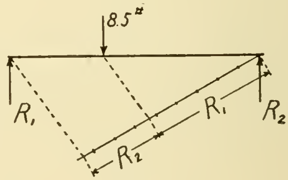


Fig. 444

In order to find such a point, draw *ef* in any convenient direction and lay off *eg* equal to $5\frac{1}{2}$ units (at any convenient scale) and *gf* equal to $2\frac{1}{2}$ units. Connect *d* and *f* and draw *gh* parallel to *df*, cutting *de* at *h*. The point *h* will establish the position of the required resultant. The amount of the resultant is 8 lbs.

Figure 444 shows a beam of known span and carrying a known load. Let the amounts of the reactions be required. The slanting line which intersects the right reaction is drawn in any

convenient position and it is made equal to the given load. Now the principle of similar triangles, used in Fig. 443, can be employed again, but it must be applied in the reverse sense, as shown in the drawing.

252. A Truss. In § 27, the free body method for the determination of stresses in a truss is developed. In that case each joint is treated as a free body and the internal stresses are determined by graphics. Our present purpose is to take up the same idea and show how the work involved in the solution given in § 27 may be shortened and organized by the use of Bows notation.*

Let Fig. 445*A* be the space diagram for a truss carrying loads as shown. The first step consists in computing the reactions (§ 41) and in noting their amounts on the drawing. Letters are now placed between each external load or reaction so that the external forces become *AB*, *BC*, . . ., *FO*, *OA*; *taken in order.* Similarly a letter is placed in each space between truss members so that each member of the truss may be named from the letters between which it occurs, as *AG*, *BJ*, . . ., *OG*, *GH*, etc.

The next step is to lay off the external forces on a ***load line*** as shown on the force diagram, Fig. 445*B*. In this diagram the external forces (including the reactions) are laid off *in order*, and in the directions given by the forces themselves as *ab*, *bc*, *cd*, *de*, *ef*, *fo*, *oa*. It is evident that these forces, being in equilibrium, must produce a closed polygon when represented on the force diagram (§ 17). *In this case* none of the external forces has a horizontal component. Therefore the force polygon becomes a straight line, but essentially it is a force *polygon* and must close up in the same way as if horizontal forces were involved. The closing of the load line indicates equilibrium between loads and reactions. We are therefore ready to proceed with the determination of stresses.

Now let the members *AG* and *GO* be cut, giving the free body as shown in the space diagram *C*.† This body is acted upon by the

* In using the *method* given in § 252, it is all too easy for the student to lose sight of the *principle* of § 27 and to substitute rules of action for real understanding. For this reason a review of §§ 20–28 will prove valuable at this time.

† Let the student be sure he understands *why* this joint is chosen as the point of beginning.

known force OA and the unknown stresses in AG and GO. A force diagram for this point is drawn at D, which determines the stresses in AG (compression) and GO (tension).

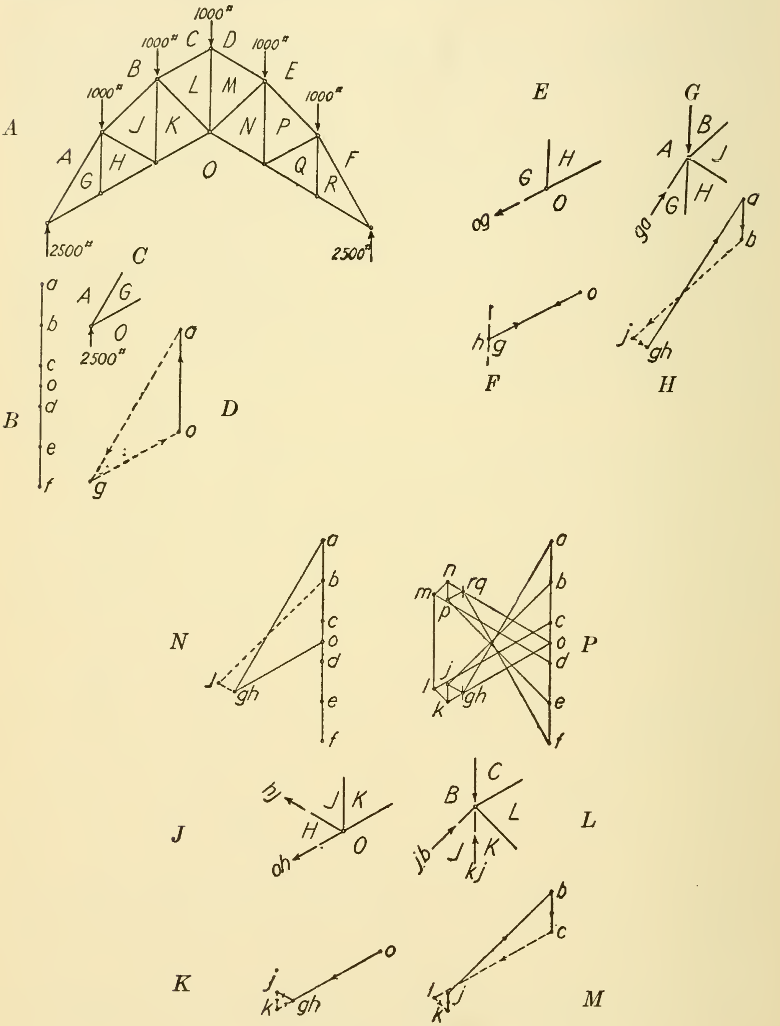


FIG. 445

The next free body, which is shown in E, cuts the members OG, GH, and HO. The stress in OG is known (from the preceding

solution) and it is laid off in the force diagram F, from o to g. A vertical line drawn through g represents the stress in GH. From the end of this line, a line drawn parallel to OH must now close back to o. This shows that h and g must coincide and hence the stress in GH is zero. This fact could have been proved from the conditions of equilibrium without drawing any force diagram. Let the student prepare a proof.

Now proceed to the next joint ($ABJHG$). The free body is shown in G. It is seen to be acted upon by the known force AB and the known stress ga. Let the force diagram for this joint be drawn at H, starting with the known stress ga and the known force ab, and closing with the unknown stresses bj (compression) and jh (tension). The points h and g coincide as shown in the preceding diagram.

In the figure, drawings J and L are space diagrams for two other joints while K and M are the corresponding force diagrams. Similar diagrams could be drawn for all the joints of the truss.*

An analysis of the load line B and the force diagrams D, F, H, K, M shows a considerable repetition of identical lines. This repetition can be avoided by attaching all the separate force diagrams to the load line. In N the load line is shown with the three force diagrams D, F, and H attached. In P the load line is shown in connection with the stresses in all members.

An examination of P shows a figure composed of lines which are controlled, *as to direction*, by the space diagram, and *as to separation* by the load line. The lengths of these lines (and hence the stresses in the truss) are thus *seen* to depend on the shape and loading of the truss.

Let us now trace the making of diagram P in detail. First the load line $a, b, \ldots f, o, a$ is laid off by taking the external forces *in order* as described above. Then taking the left reaction joint as a free body, we trace out its force diagram, oa, ag, go. We start with the known force *oa and trace it from o to a in the direction given by the known reaction.* From a we continue along ag and go and thus we arrive back at our starting point, o. In doing this

* The student is advised to do this before proceeding.

we have established the directions of ag and go and hence the fact that AG is in compression and GO in tension. *

Since the stress in GH is zero (as already shown), we can at once establish the point h on the force diagram as coinciding with g, or we can trace out a polygon similar to F, which would give the same result.

Next take the joint $ABJHG$ as a free body. The stress in AG and the load AB are known and already drawn on the force diagram P. Starting with ga and ab (the direction being given by the known force directions), we trace out the polygon ga, ab, bj, jg. For the next joint the polygon is oh, hj, jk, ko. Then comes bc, cl, lk, kj, jb. At the peak we obtain lc, cd, dm, ml.

The diagram which has just been built up is called the ***connected stress diagram.*** It contains no information which cannot be found on the separate stress diagrams D, F, H, K, M. In fact it is nothing more than an aggregation of the diagrams for individual joints. The only reason for putting stress diagrams in this connected form lies in the matters of convenience and methodical procedure.

The basis of the connected stress diagram is *order*. Given a load line on which all the external forces (including the reactions) have been laid off in a certain order, the individual stress diagrams should be built up on the load line in an order based on the one first chosen. In this way it is possible to organize the procedure in a very effective manner. It would be easy to give a *rule* for doing this work, but it is far better for the student to derive his own rules of procedure through practice.

253. A Cantilever Truss. In Fig. 446A is shown another example somewhat like the previous case but more complex as to reactions and loading. The first step is to compute the reactions, CD, DE, and EO. The load line is then laid off from o to a, b, c, d, e and back to o. Thus the forces are taken *in the order and direction* of their occurrence. The stress diagram, Fig. 446B, can now be traced out just as before.

* If the student has difficulty in following this idea, let him refer back to §§ 23–28.

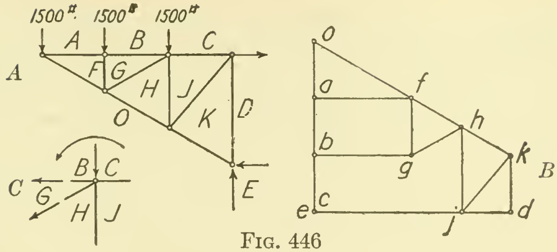


FIG. 446

254. General Principles of Procedure. The two preceding cases illustrate special applications of Bow's notation to the graphic solution of stresses in a truss or similar frame. It is perhaps worth while to point out a few general principles and methods which will facilitate progress.

(*a*) *The reactions* must be determined accurately. This is *vital* to the solution. If the reactions are incorrectly determined, the truss as a whole is not in equilibrium (ΣM does not equal zero) and therefore the laws of equilibrium cannot be applied in solving for the stresses. In §§ 259–261 a graphic method for determining reactions is given.

(*b*) *The load line* is a graphic summation of the external forces. It must be a closed polygon. Otherwise the conditions of equilibrium ($\Sigma H = 0$, $\Sigma V = 0$) are not satisfied. It is useless to proceed with a solution for stresses until after the reactions and load line have been determined *and checked.*

(*c*) *Order* is the essence of the problem in the use of *Bow's notation.* The load line is constructed by laying off the external forces *in the same order* as that in which they occur on the space diagram, and in the directions indicated by the character of the forces themselves. The load line may be started with any force and continued by the forces either to the left or to the right of the one first chosen. But the *order first chosen* and the proper directions must be maintained carefully. It is usual to start with the load lying farthest to the left and that method will be followed in this chapter.

(*d*) *An established procedure* is important. As an illustration of one good method let us take the case shown in Fig. 446*A*, and

let us trace out the stress diagram for the joint *GBCJH*. The stresses in *HG* and *GB* are known; also the load *BC*, as shown in Fig. 446*C*. In starting to trace the stress diagram, it is best to circle the joint (as indicated by the arrow) until we arrive at a member containing an unknown stress, as *HJ*. Now starting with the letter to the *left* of that member, as *H*, trace out the *known* stresses and forces *in order* on the force diagram, as *hg*, *gb*, *bc*. This establishes the direction in which the polygon reads and from *c* we can draw *cj* and from *h* we can draw *hj*, thus closing the polygon. The characters of the unknown stresses are then fixed by the known direction already established, i.e., *c* to *j* (showing tension in *CJ*) and *j* to *h* (showing compression in *HJ*).

(*c*) *The self-checking* feature of these diagrams should not be overlooked. Thus in Fig. 446*B*, the point *k* is the one last determined. The line *ko* should then be found to be parallel to *KO* in the space diagram. This constitutes a good check on the correctness of the work.

255. Resultant of Several Non-Concurrent Forces. A. General. The resultant of several forces can be found by the graphic method given in § 38 if the forces are not parallel. If the forces are parallel, the position of the resultant can be found by the analytic method given in §§ 41–44. When many forces are

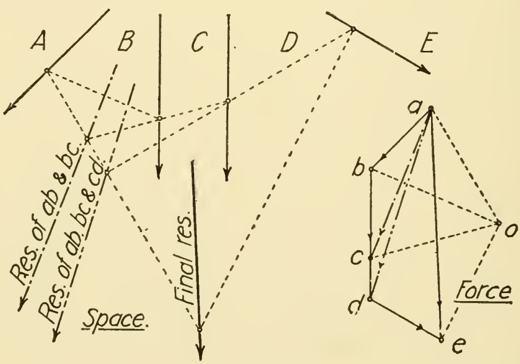


Fig. 447

to be combined, these methods become excessively tedious and cumbersome. This is especially true of the graphic method of

§ 38. The intersections frequently occur at points so remote from the main part of the drawing as to call for drawing boards and straight edges of quite impossible dimensions. Therefore, we give below a method which employs a similar principle but which is simpler in execution.

B. THE EQUILIBRIUM POLYGON. In Fig. 447, let it be required to find the amount and position of the resultant of the forces $AB \ldots DE$. From §§ 15–16, we know that the *amount* and *direction* of the required resultant may be found by laying off the forces continuously to scale. This has been done in the force diagram, Fig. 447, where *abcde* is the load line for the given forces and *ae* gives the *amount* and *direction* of the resultant.

The *position* of the resultant must now be determined. Since position is a space relation, it will of course involve the use of the space diagram. In order to study this in detail, let the first two forces be drawn separately as in Fig. 448. Whatever relation, whether of equilibrium or lack of equilibrium, may exist between AB and BC, that relation will not be disturbed if we introduce into the system two forces which are equal and opposite and which act in the same straight line.*

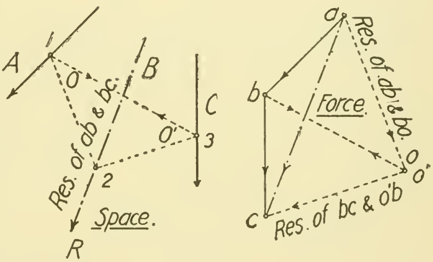


FIG. 448

Let us then arbitrarily introduce into the problem two forces. *bo* and *o'b*, which are equal and opposite (as shown on the force diagram) and which act in the same straight line (as shown by *1–3* on the space diagram). Now *ab* and *bo* have a resultant, *ao*, and this resultant acts through the point *1* (on the space diagram). Similarly *o'b* and *bc* have a resultant which acts at the point *3*. These two resultants have now replaced the two original forces

* This idea of arbitrarily introducing balanced elements into a problem is not an uncommon one. The neutral elements do not affect the result but do simplify the solution. One case in point is the addition of a quantity to each side of a quadratic equation to complete the square.

and the two ***introduced forces.*** Again these two forces, *ao* and *o′c*, have a resultant which is given in amount and direction on the force diagram by *ac*; and whose position is determined on the space diagram by the intersection of *1–2* and *2–3*.

In Fig. 449, the first *three* forces of Fig. 447 have been drawn and the operations performed in Fig. 448 have been shown by dotted lines. The resultant of *AB* and *BC* is lettered *R*. We may now repeat the operation performed in Fig. 448, using *R* and *CD* as the given forces and letting *oc* represent the pair of forces introduced between them. These forces are introduced on the line *3–4* in the space diagram. Now *R* and *co* have the resultant *ao* (force diagram), which acts through *3* (space diagram) and which is shown by *3–5*. Also *oc* and *cd* have a resultant *od* which acts through *4* and which is shown by *4–5*. The final resultant is *R′*, whose amount and direction must evidently be given by a line connecting *a* and *d* on the force diagram, and whose position is given by the point *5* on the space diagram. Evidently this operation can be repeated as often as necessary to account for all of the forces in the problem.

Before going further it will be wise to make some observations and definitions. First as to the force diagram. The location of the point *o* is dependent on the amount and direction of the first pair of introduced forces. Since these forces are chosen

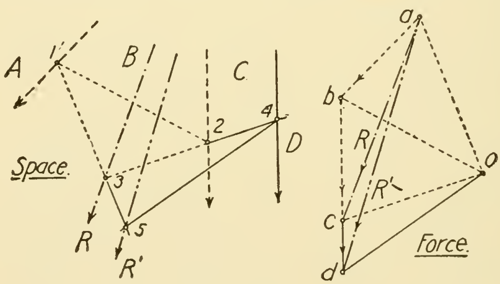


FIG. 449

arbitrarily, the point *o* (called the ***pole***) may be anywhere. The line *abcd* is called the ***load line*** and the lines from the pole to the load line are called ***rays.*** Now the resultant of all the forces in the load line is always equal to the resultant of the two outside

rays, *ao* and *od*, Fig. 449. This resultant is represented by a line joining the ends of the load line *ad*. On the space diagram we have a polygon *1–2–4–5–1*, each line of which (called a ***string***) is parallel to a ray in the force diagram. This polygon is started from a point (*1*) chosen at random, just as *o* was chosen at random in the orce diagram. The two outside strings of this polygon (*1–5* and *4–5*) represent the position of the two outside rays (*ao* and *do*) of the force diagram. Therefore, their point of intersection, *5*, gives the position of the final resultant. The polygon *5–1–2–4–5* is called the ***equilibrium polygon.***

In Fig. 447 the same principles are applied to finding the resultant of four forces.

256. Equilibrium Polygon.—Forces in Equilibrium. We have proved already (§ 17) that if a system of forces is in equilibrium the force polygon (load line) must be closed. This is equivalent to the analytic conditions of equilibrium which are expressed by $\Sigma H = 0$ and $\Sigma V = 0$. We now will show that if a system of forces is in equilibrium the equilibrium polygon also must be closed. This corresponds to the analytic condition expressed by $\Sigma M = 0$.

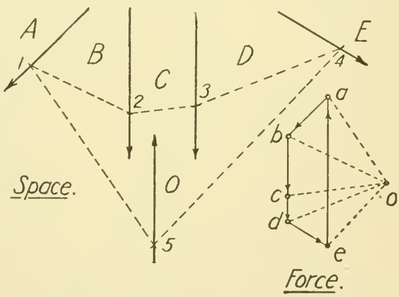


Fig. 450

In Fig. 450 we have five forces in equilibrium. The first four, *ab* . . . *de*, are the same forces shown in Fig. 447. The equilibrium polygon *5–1–2–3–4–5* is also the same as in Fig. 447 and the outside lines *5–1* and *4–5*, meeting at *5*, give the position of the resultant in Fig. 447. But the equilibrant of these five forces (*ea*, Fig. 450) must be equal and opposite to the resultant, *and it must act in the same straight line as the resultant.* Hence, equilibrium will exist in Fig. 450 if, and only if, the last force, *ea*, is found acting through the point *5*. Let the student move the vertical force to the right or left of the position given in Fig. 450 and then let him trace the resulting equilibrium polygon.

These considerations may be summed up as follows:

$$\text{If} \left\{ \begin{array}{l} \Sigma H = 0 \\ \Sigma V = 0 \end{array} \right\}, \text{ the force polygon is closed.}$$

$$\text{If } \Sigma M = 0, \quad \text{the equilibrium polygon is closed.}$$

257. The Equilibrium Polygon as a Frame Diagram. In Fig. 451 we have three forces, *ab*, *bc*, and *cd*. Let an equilibrium polygon *4–1–2–3–5* be drawn. At each of the points *1*, *2*, and *3* we now have three forces acting; and each group is in equilibrium in itself as shown by drawings *B* and *D*. Moreover the two forces acting in the line *BO* are equal and opposite as are also the two forces acting on the line *CO*. Therefore the whole system is in equilibrium so long as the forces *OA* and *DO* are applied where shown in drawing *B* and are of the amounts shown in drawing *D*.

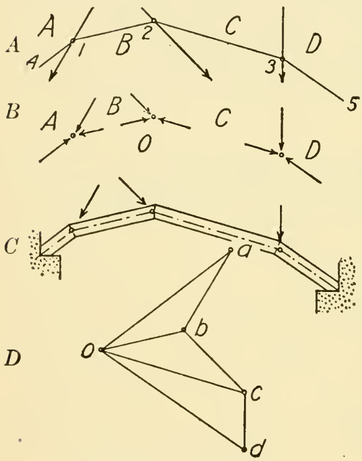


Fig. 451

Now let us imagine that the forces *oa* and *do* (Fig. 451*D*) are applied at the points *4* and *5* (Fig. 451*A*); then evidently the entire system of forces is in equilibrium. Moreover each line of the equilibrium polygon is the line of action of two equal and opposite forces, thrusting towards the ends of the line. Therefore, if a timber frame be built, taking its form from the equilibrium polygon as shown in Fig. 451*C*, and if it be loaded with loads as shown, and provided with proper reaction points (corresponding to *4* and *5* in drawing *A*), the frame would be in equilibrium under the given loads. Also it would be stressed in compression in each of its parts. The amounts of these compressive stresses are given by the corresponding rays of the force polygon. Thus we see that every equilibrium polygon gives the form of a frame which can

carry the given loads without any tendency to rotate or distort. It is well to notice that

(*a*) The reactions must be provided somewhere on the two outside lines of the equilibrium polygon.

(*b*) There is a different frame diagram for each different set of loads and for each position of the pole chosen for the force polygon.

(*c*) If the pole is selected as in Fig. 451*B*, the stresses in the frame are compression. If it is chosen on the opposite side of the load line, the stresses are tension.

(*d*) Selecting a pole close to the load line gives a deep frame with relatively small reactions, and vice versa.

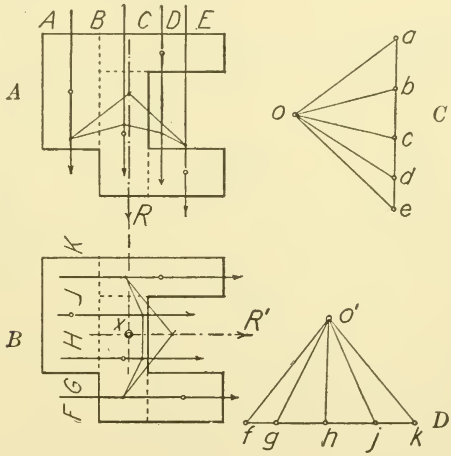
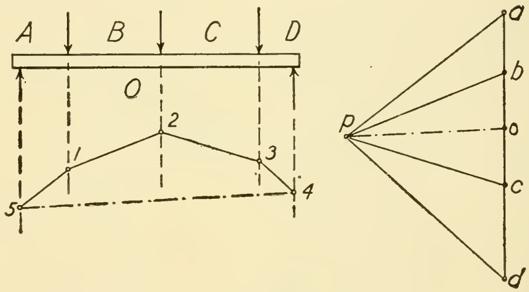


Fig. 452

258. Center of Gravity. In Fig. 452*A* the area shown is divided into four parts. The center of gravity of each part is found by inspection and the lines *AB*, *BC*, *CD*, *DE* are drawn to represent the lines of action of the (nominal) weights of the parts. The force polygon *C* is laid out with the lines *ab* . . . *de* made proportional to the areas of the corresponding parts of Fig. 452*A*. Now when we draw the corresponding equilibrium polygon, the resultant *R* is found. The center of gravity of the area lies on this line. In Fig. 452*B*, by the same process as above,

the center of gravity is found to lie on the line R'. Then by superimposing R (from Fig. 452A) on this diagram we get the point x, which is the center of gravity of the area.

259. Reactions of a Beam.—All Forces Vertical. In Fig. 453A let the beam and the loads be given and let it be required to find the reactions. In the force diagram, Fig. 453B, the loads ab, bc, cd are laid off to scale. Since all the forces are vertical, it is evident that the sum of the reactions must be equal and opposite to ad. When these reactions are laid off *in order*, with the forces, the load line will read ab, bc, cd and from d upward to some *unknown* point o and from o to a, thus making the load line a closed polygon. It remains to locate o.



A FIG. 453 *B*

From any point, *1*, on the line of action of AB, as a point of beginning, lay off an equilibrium polygon cutting the lines of the known reactions at *4* and *5*. Now the three forces and the two reactions are known to be in equilibrium. Hence their equilibrium polygon must close (§ 256). Therefore let us draw the line *4–5* to close the equilibrium polygon. Now there must be a ray in the force polygon to correspond to the string *4–5*, just drawn. Draw such a ray, po, parallel to *4–5*. The point o thus found determines the reactions; do being the amount of the right reaction while oa is the amount of the left reaction.

260. Bending Moment Diagrams. An equilibrium polygon sometimes may be interpreted as a bending moment diagram. This may be seen by reference to Fig. 454. A beam is given, loaded with two loads AB and BC. By means of an equilibrium

polygon, the reactions *co* and *oa* are determined (§ 259). Now imagine the beam to be cut at the section *SS*. The bending moment at that section is equal to the moment of all forces to the left (or right) of the section, i.e., it is equal to the moment of the forces *OA* and *AB* (or what is the same thing it is equal to the moment of the *resultant* of *OA* and *AB*) taken about the section *SS* as a center. On the force diagram the resultant of *oa* and *ab* is seen to be *ob*. From the equilibrium polygon already drawn, the position of this resultant is found (as in § 255) by continuing lines *1–2* and *3–4*, till they meet at o'.

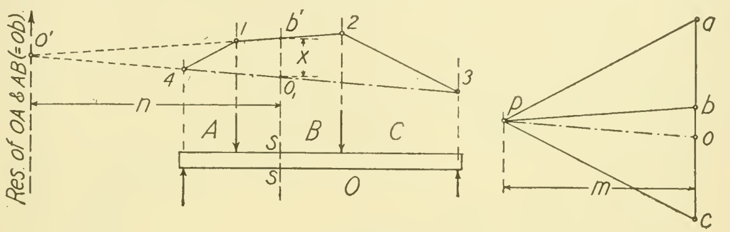


FIG. 454

This resultant, being equal to *ob* lbs., and having a lever arm about the section *SS* of *n* inches, produces a bending moment of $(ob)n$ lb. ins. The triangles $o'b'o_1$ (on the space diagram) and *pbo* (on the force diagram) are similar, therefore

$$x : ob = n : m,$$
$$(ob)n = xm.$$

But we have shown that $(ob)n$ is the bending moment at the section *SS*. The vertical distance x is measured across the equilibrium polygon at *SS* and m is the horizontal component of the introduced forces represented by the rays of the force polygon. From the preceding discussion it becomes clear that the bending moment (in pound inches) at any section cut through a beam is equal to the vertical distance across the equilibrium polygon at that section (measured in inches) multiplied by the horizontal component of the rays in the force polygon (measured in pounds). Let the student now repeat the above demonstration using a different case which involves a larger number of loads.

261. Wind-Load Reactions. Wind blowing on a roof produces loads on a truss which have horizontal components. In such a case the reactions cannot be obtained as in § 259. In fact the case is statically indeterminate (as explained in § 40) unless (as is sometimes done) a roller is placed under one end of the truss. The effect of the roller is to make the reaction at that end take a definitely vertical direction and thus to reduce the unknown components to three.

A. Roller under One End. (1) The reactions may be found by first finding the resultant of all the loads acting on the truss. This can be done by using an equilibrium polygon as in § 255 or by analytic methods. All forces on the truss are now reduced to three and these three must meet at a point (§ 39). From this point the solution follows (§ 16).

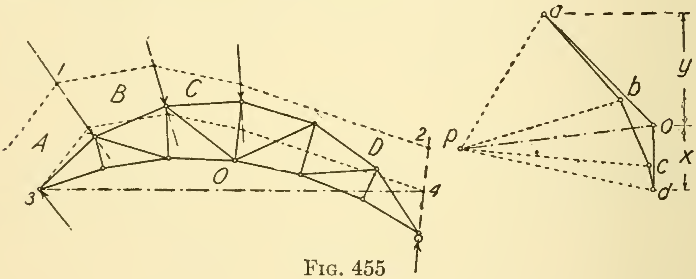


Fig. 455

(2) Another solution for this case is illustrated in Fig. 455. The loads *AB*, *BC*, and *CD* are given; the reaction *DO* is known to be vertical; while *OA* is wholly unknown. Draw the load line *abcd* and from *d* draw a vertical line. The reaction *do* will follow this line to some (unknown) point *o* and from there the other reaction (*oa*) must close back to *a*.

In order to find *o* let us attempt to draw an equilibrium polygon starting at point *1*. If we follow the method of § 259, this polygon will close up on the point *2* on the right reaction and at some other point on the left reaction. But since the left reaction is unknown in direction, we cannot obtain the needed closing line. However we know that the left reaction (whatever may be its direction) must pass through the point *3*. Therefore if we slip our polygon down until the outside string passes through

the reaction point *3*, it will then have both ends resting on the reaction lines and a closing line can be drawn. Such a polygon is shown in the figure, drawn in dotted lines and closed by the line *3–4*. We can now draw the last ray of the force polygon, *po*, stopping it on the vertical previously drawn through *d*. The reactions are now determined as *do* and *oa*.

B. Both Ends Fixed in Place. When both ends of the truss are fixed in place the reactions are indeterminate, as explained in § 40. There are however two commonly used *assumptions* that may be made the basis of a statical solution.

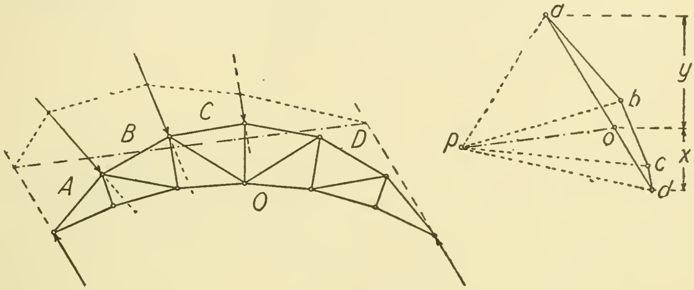


Fig. 456

(1) *The reactions are assumed to be parallel.* In Fig. 456 the load line *abcd* calls for reactions that will close up from *d* to *a*. If these reactions are parallel they will form a continuous line as *da*, leaving their separation into amounts to be determined by the unknown point *o*.

The lines of the reactions can now be drawn on the space diagram and the solution completed precisely as in § 259.

(2) *The reactions are assumed to have equal horizontal components.* In this case we do not know the direction of either reaction. But we do know one point on each line, namely, the two reaction points. It will be our purpose to draw an equilibrium polygon that will pass through these two points. In Fig. 457, draw the load line *abcd* as usual and, using the pole *p*, find the position of the resultant load *R*, as shown.

Now choose any point on *R*, as *3*, and draw lines from it to the reaction points *1* and *2*. Now the lines *1–3*, *2–3* constitute an equilibrium polygon for the force *R*, the closing line of which is

1–2. Draw the corresponding rays on the force polygon, *an* and *nd*, thus locating the pole of the force polygon which corresponds to the equilibrium polygon *1–3–2*. From *n* draw a line parallel to *1–2*, being the ray corresponding to the closing line of the equi-

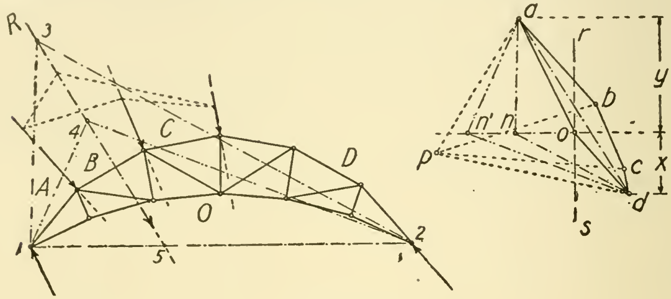


Fig. 457

librium polygon. The reaction lines must meet on this line. Also (from the assumption) the reaction lines must fall on a line midway (in a horizontal sense) between *a* and *d*, i.e., on the line *rs*. This locates the point *o* as the point where the reaction lines must meet and determines the reactions as *do* (right) and *oa* (left).

If some point on *R* other than *3* had been chosen, from which to construct the equilibrium polygon for the force *R*, the result would have been the same. Thus, choosing *4*, the equilibrium polygon becomes *1–4–2*. The pole of the force polygon is found at *n′* and the reactions are as before. In other words, for every point on *R* which may be chosen as the vertex of the equilibrium polygon, the pole of the corresponding force polygon will be found on the horizontal line *on′*. This can be proved analytically by extending the force *R* to the point *5* and treating its *H* and *V* components separately. Let the student prepare the proof.

C. Choice of Assumptions. The results obtained from the two assumptions explained in (A) and (B) do not differ materially *in the ordinary case*. Where the supports are equally stiff and the truss itself is reasonably stiff, there is perhaps more to justify the assumption of equal horizontal components. However, one should not forget that in any case (except when a roller or other similar device is used) the problem is statically indeterminate.

NOTATION

The following is a list of the symbols used in the text. In so far as possible these symbols agree with the list approved by the Society for the Promotion of Engineering Education, June 28, 1918, and published in the Bulletin of the Society, November, 1923. Symbols which are used but not included in that list are marked *. In the case of Reinforced Concrete a separate set of symbols which has become standardized in the literature of the subject has been used. See page 296.

* Angles—in general, Greek letters, e.g.	$\alpha \beta \theta \phi$
Area	A
Breadth	b
* Center of gravity—distance to	$\bar{x}$, $\bar{y}$ or $\bar{z}$
Center of rotation	ϕ
* Coefficient of linear expansion	c
Coefficient of friction	f
Coefficients and constants	C
* Concrete—reinforced	See p. 296.
Definite integral—See footnote p. 54	$\int_W B\, dw$
Deflection of a beam	y
* Deformation—linear	q
Depth	d
Diameter	D
Distance of extreme fibre from the neutral axis	c
Eccentricity of application of load	e
* Elasticity, modulus of	E
* Forces—known	A, B, C, etc.
* Forces—unknown	X, Y, Z, etc.
Force—moment of	M
Friction, coefficient of	f
* Gyration—radius of	r
Height	h
Inertia, rectangular moment of	I
* Inertia, product of	K
Length	L
* Load, concentrated (e.g., on a beam)	P
* Load, distributed (e.g., on a beam), in pounds per linear unit	w
* Load, distributed (e.g., on a beam), total	W
* Load, direct tension, compression or shear	(See Force)
Load, eccentricity of	e
* Modulus of section	$\frac{I}{c}$

* Modulus of elasticity . . . . E
Moment of force . . . . M
Moment of inertia—rectangular I
* Product of inertia . . . . K
* Radius of gyration . . . . r
Reactions . . . . R
Reinforced concrete . . . . See p. 296.
* Section modulus . . . . $\frac{I}{c}$
* Shear—total in a beam . . . . J
* Slope—of the elastic curve in a beam v
* Span—of a beam L
* Stress, unit—in general . . . . s
* Stress, unit—in tension, compression, shear or flexure . . . . s_t, s_c, s_s, s_f
* Stress, total . . . . S
* Temperature—change of . . . . T
* Thickness . . . . t
Volume V
Weight . . . . W

APPENDIX

TABLES

The tables which follow give a number of the constants needed in the solution of the problems. These tables are of necessity much condensed and are not intended for general reference, but only to give a basis for obtaining comparable answers to the problems.

No such condensed tables can do more than give very roughly approximate information. For instance, in Table I the ultimate compressive strength of timber is given as 3,000 lbs. per sq. in., whereas in fact the various kinds of timber under various conditions show values ranging all the way from 1,800 lbs. per sq. in. to 5,200 lbs. per sq. in. Again, in the case of the metals, slight variations in the chemical composition or treatment during manufacture may cause equally large or larger variations in the physical properties.

Therefore, these tables should not be used for computations which are to form the basis of actual construction. Information for such purpose should be sought in books of reference and in the light of the special conditions of the actual problem. However, the condensed form of the tables will be found a good one for the beginner's use and the values quoted may ultimately remain in his mind, useful for the purpose of arriving at quick approximations.

TABLE 1

THE PHYSICAL PROPERTIES OF VARIOUS MATERIALS

Material	Values given in thousands of lbs./sq. in.									Modulus of elasticity values given in millions of lbs./sq. in.	Coefficient of expansion for 1° F.	Weight in lbs./cu. ft.
	Ultimate strength				Working strength				Elastic limit[1]			
	Tens.	Comp.	Shear.	Bend.[7]	Tens.	Comp.	Shear.	Bend.[7]				
Aluminum	15.00	12.00	12.00	13.00	3.30	3.00	3.00	3.25	9.00	9.00	.0000128	165
Brass	20.00	30.00	36.00	24.00	5.00	7.50	9.00	6.00	10.00	11.00	.0000104	534
Brick	0.20	10.00	0.40	0.60	0.02	1.00	0.04	0.06	[3]	2.00[4]		
Brick Masonry	[5]	2.00	[5]	[5]	0	0.20	0	0	[3]	[3]	.0000031	120
Bronze	30.00	50.00	25.00	34.00	7.50	12.50	6.25	8.50	20.00	13.00	.0000101	509
Cast Iron	25.00	80.00	20.00	30.00	5.00	16.00	4.00	8.00	15.00[6]	12.00[4]	.0000059	450
Concrete	0.20	2.00	1.00	0.35	0.02	0.45	0.10	[5]	[3]	2.00[4]	.0000067	144
Copper	30.00	35.00	20.00	30.00	5.00	8.00	4.50	7.50	10.00	10.00	.0000093	556
Glass	3.00	30.00								8.00	.0000047	156
Lead	1.75								[3]		.0000159	710
Plaster	0.07	0.70									.0000092	120
Steel	60.00	60.00	45.00	60.00	16.00	16.00	12.00	16.00	30.00	29.00	.0000067	490
Stone	[5]	10.00	2.00	2.50	0	1.00	0.20	0.25	[3]	7.5[4]	.0000060	160
Stone Masonry		2.50		1.50	0	0.25		0.15	[3]	[3]	.0000035	145
Timber—forces parallel to the grain	12.00	3.00	0.30	5.00	3.00	0.80	0.08	1.00	[6]	1.00	.0000030	40
Timber—forces perpendicular to the grain					0.06	0.20	0.40				.0000250	40
Wrought Iron	48.00	48.00	40.00	48.00	12.00	12.00	8.00	12.00	26.00	27.00	.0000067	485
Zinc	6.00	20.00	17.00	10.00					[3]	12.00	.0000173	440

[1] Elastic limits are given for tension except where otherwise noted.
[2] Modulus of elasticity is the same for tension and compression except when noted.
[3] Elastic qualities poorly defined or non-existent. See § 73.
[4] Value given is for compression. The value in tension is poorly defined or non-existent.
[5] Strength very low and uncertain.
[6] Elastic limit in compression is close to the ultimate strength.
[7] See § 137.

TABLE II

Factors of Safety

Note. Tables of factors of safety are rarely found in technical literature. This is probably due to a natural reluctance on the part of the writer to commit himself on a subject that is very involved and in which it is most difficult to guard against numerous possible misunderstandings.

The following is taken from Merriman's *Mechanics of Materials.* It was intended there merely as a basis for discussing the problems in the text and it is here repeated for the same purpose. No such simple table could possibly cover all of the contingencies mentioned in § 61. It is at least doubtful whether any possible table could properly cover the subject. (See footnote, page 95, also § 62.)

Usual Factors of Safety (on the Basis of Ultimate Strength)

Material	Steady stress	Variable stress	Shocks
Brick and stone	15	25	40
Timber	8	16	25
Cast iron	6	10	20
Wrought iron	4	6	10
Steel—structural	4	6	10
Steel—hard	5	8	15

TABLE III

Notes and Explanation. On the following pages, eighteen cases of beam loadings are summarized. The load, shear, moment, slope, and deflection curves are drawn and the equations of the curves are stated. In general, maximum and minimum values are determined and located.

In most of the cases the origin of coordinates is taken at the left end of the beam; but in special cases, such as No. 5 and No. 18, more than one origin is used.

Forces directed upward are treated as positive. Distances measured upward or to the right are positive. The slope of a curve is positive when the curve slants upward and to the right or downward and to the left. In the diagrams, the loading is indicated above the beam in order to maintain a semblance of reality, But since the loading is a force acting downward, it must be treated as a negative quantity.

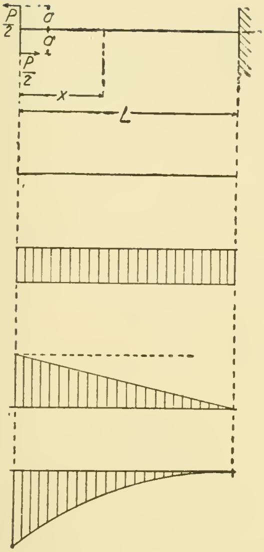
The purpose of the table is two-fold: first, to serve as reference material on the critical values of shear, moment, and deflection covering a larger number of cases than can be derived profitably in the text; second, to bring out more fully the manner in which any given curve is related to the curves above and below it (§ 156).

The general equations, on which the equations for the special cases are based, can be found in §§ 121, 123, 124, and 152. In the curves for slope and deflection, E and I are treated as constants. This means that the equations apply to beams of homogeneous material and constant cross section only.

In the equations, the symbols V and M indicate the shear and bending moment. In the curves and equations covering slope and deflection, the results have been given in terms of EIv and EIy. Thus the right-hand member of any equation shows the effect of load and span only. When a given loading gives rise to two or more separate curves, as in No. 2, No. 5, No. 12, etc., the beam is divided into sections marked I, II, III, etc. The shear, moment, slope, and deflection for section I are indicated as usual by V, M, EIv, and EIy. For section II the same quantities are primed, V', M', etc. For section III we use V'', M'', etc.

The equations for two or more cases of loading may be combined to give the effects of more complex loadings, as explained in §§ 126 and 158.

TABLE III



Vertical Loading 1

$W = 0$

Shear

$V = 0$

Moment

$M = -Pa$

Slope

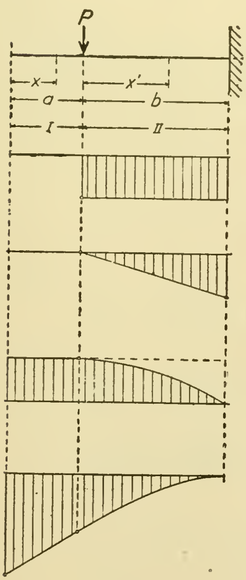
$EIv = M(x - L)$

EIv (max.) $= -ML$, at the free end.

Deflection

$$EIy = -\frac{M}{2}(L - x)^2$$

$$EIy(\text{max.}) = -\frac{ML^2}{2}, \text{ at the free end.}$$



Load $= P$ 2

Shear

I, $V = 0$

II, $V' = -P$

Moment

I, $M = 0$

II, $M' = -Px'$

$M'(\text{max.}) = -Pb$, at the fixed end.

Slope

$$\text{I, } EIv = \frac{Pb^2}{2}$$

$$\text{II, } EIv' = \frac{P}{2}(b^2 - x'^2)$$

Deflection

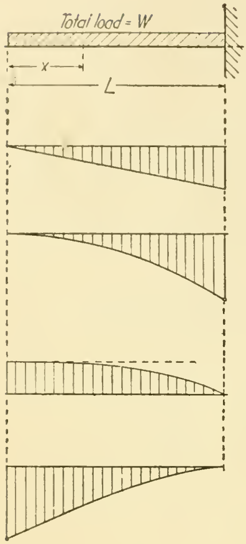
$$\text{I, } EIy = \frac{Pb^2}{6}(3x - 3a - 2b)$$

$$EIy(\text{max.}) = -\frac{Pb^2}{6}(2b + 3a), \text{ at the free end.}$$

$$\text{II, } EIy' = -\frac{P}{6}(x'^3 - 3b^2x' + 2b^3)$$

$$EIy'(\text{max.}) = -\frac{Pb^3}{3}, \text{ at the load.}$$

TABLE III, *continued*



Load 3

Rate of Loading $= -\dfrac{W}{L}$ lbs. per ft.

Total Load $= W$

Shear

$$V = -\frac{W}{L}x$$

Moment

$$M = -\frac{W}{2L}x^2$$

$M(\text{max.}) = -\dfrac{WL}{2}$, at the fixed end.

Slope

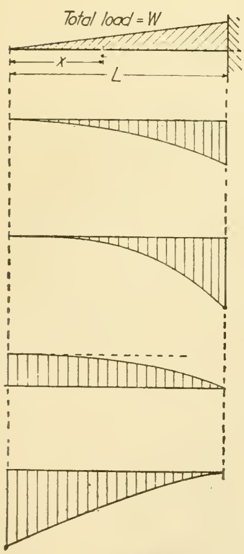
$$EIv = \frac{W}{6}\left(L^2 - \frac{x^3}{L}\right)$$

$EIv(\text{max.}) = \dfrac{WL^2}{6}$, at the free end.

Deflection

$$EIy = -\frac{W}{24}\left(\frac{x^4}{L} - 4L^2x + 3L^3\right)$$

$EIy(\text{max.}) = -\dfrac{WL^3}{8}$, at the free end.



Loading 4

Total Load $= W$.

Rate of Loading: at any point $= -\dfrac{2W}{L^2}x$

at the fixed end $= -\dfrac{2W}{L}$

Shear

$$V = -\frac{W}{L^2}x^2$$

Moment

$$M = -\frac{W}{3L^2}x^3$$

$M(\text{max.}) = -\dfrac{WL}{3}$, at the fixed end.

Slope

$$EIv = \frac{W}{12}\left(L^2 - \frac{x^4}{L^2}\right)$$

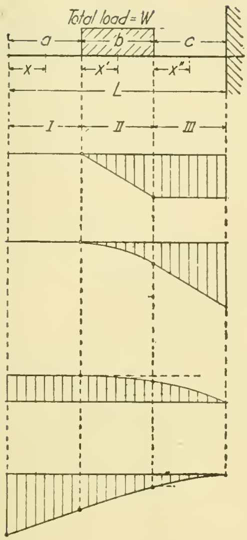
$EIv(\text{max.}) = \dfrac{WL^2}{12}$, at the free end.

Deflection

$$EIy = -\frac{W}{60}\left(\frac{x^5}{L^2} + 4L^3 - 5L^2x\right)$$

$EIy(\text{max.}) = -\dfrac{WL^3}{15}$, at the free end.

TABLE III, *continued*



Load **5**

Rate of Loading $= -\dfrac{W}{b}$ lbs. per ft., on section II

Total Load $= W$.

Shear

I, $V = 0$

II, $V' = -\dfrac{W}{b}x'$

III, $V'' = -W$

Moment

I, $M = 0$

II, $M' = -\dfrac{Wx'^2}{2b}$

III, $M'' = -W\left(x'' + \dfrac{b}{2}\right)$

$M(\text{max.}) = -W\left(c + \dfrac{b}{2}\right)$, at the fixed end.

Slope

I, $EIv = \dfrac{W}{2}\left(c^2 + cb + \dfrac{b^2}{3}\right)$

II, $EIv' = \dfrac{W}{2}\left(c^2 + cb + \dfrac{b^2}{3} - \dfrac{x'^3}{3b}\right)$

III, $EIv'' = \dfrac{W}{2}(c^2 + cb - x''^2 - bx'')$

EIv(max.) occurs throughout space I.

Deflection

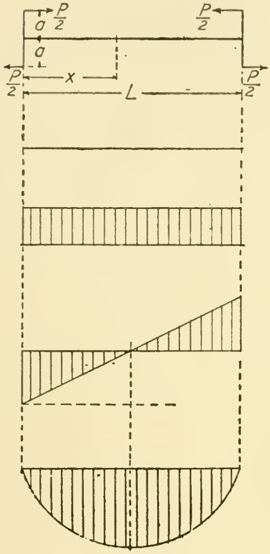
I, $EIy = -\dfrac{W}{24}[(3c^2 + 3cb + b^2)(3c + 3b + 4a - 4x) - c^3]$

II, $EIy' = -\dfrac{W}{24}(b - x')\left[4(3c^2 + 3cb + b^2) - \dfrac{(x' + b)(x'^2 + b^2)}{b}\right]$

III, $EIy'' = -\dfrac{W}{12}(c - x'')^2(2x'' + 4c + 3b)$

EIy(max.) can be found by putting $x = 0$ in I.

TABLE III, *continued*



Vertical Loading **6**

$W = 0$

Shear

$V = 0$

Moment

$M = Pa$

Slope

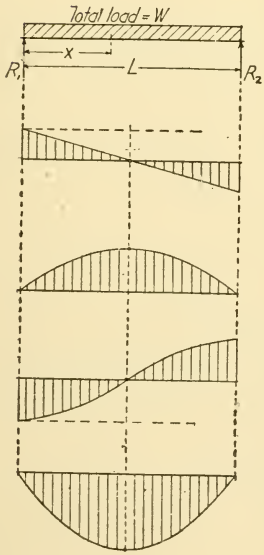
$$EIv = M\left(x - \frac{L}{2}\right)$$

$$EIv(\text{max.}) = \pm M\frac{L}{2}, \text{ at the ends.}$$

Deflection

$$EIy = -\frac{M}{2}(Lx - x^2)$$

$$EIy(\text{max.}) = -\frac{ML^2}{8}, \text{ at the center.}$$



Load **7**

Rate of Loading $= -\frac{W}{L}$ lbs. per ft

Total Load $= W$.

$R_1 = R_2 = \frac{W}{2}$.

Shear

$$V = \frac{W}{2} - \frac{Wx}{L}$$

Moment

$$M = \frac{W}{2}\left(x - \frac{x^2}{L}\right)$$

$$M(\text{max.}) = \frac{WL}{8}, \text{ at the center.}$$

Slope

$$EIv = \frac{W}{2}\left(\frac{x^2}{2} - \frac{x^3}{3L} - \frac{L^2}{12}\right)$$

$$EIv(\text{max.}) = \pm\frac{WL^2}{24}, \text{ at the supports.}$$

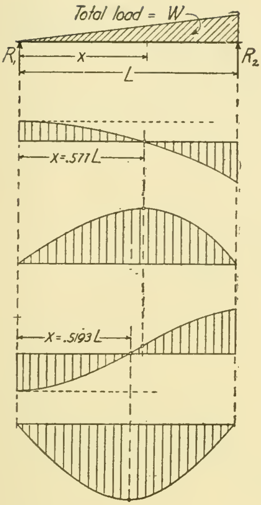
Deflection

$$EIy = \frac{W}{24}\left(2x^3 - \frac{x^4}{L} - L^2x\right)$$

$$EIy(\text{max.}) = -\frac{5\ WL^3}{384}, \text{ at the center.}$$

TABLE III, *continued*

8



Loading

Total Load $= W$.

Rate of Loading: at any point $= -\dfrac{2W}{L^2}x$

at the right end $= -\dfrac{2W}{L}$

$$R_1 = \frac{W}{3} \qquad R_2 = \frac{2W}{3}$$

Shear

$$V = \frac{W}{3L^2}(L^2 - 3x^2)$$

$V = 0$ when $x = L\sqrt{\tfrac{1}{3}} = .577L$

Moment

$$M = \frac{W}{3L^2}(L^2x - x^3)$$

$$M(\text{max.}) = \frac{2WL}{9\sqrt{3}} = .1283\,WL,$$

when $x = .577L$.

Slope

$$EIv = \frac{W}{6}\left(x^2 - \frac{x^4}{2L^2} - \frac{7L^2}{30}\right)$$

$EIv = 0$, when $x = .5193L$.

Deflection

$$EIy = -\frac{W}{6}\left(\frac{x^5}{10L^2} - \frac{x^3}{3} + \frac{7L^2x}{30}\right)$$

$EIy(\text{max.}) = .01304WL^3$, when $x = .5193L$

9

Note.—The loading conditions in this case are symmetrical. Therefore all equations are given for the left half of the beam only.

Figure
(See next page)

Loading

Total Load $= W$

Rate of Loading; at any point $= -\dfrac{4Wx}{L^2}$

at the center $= -\dfrac{2W}{L}$

$$R_1 = R_2 = \frac{W}{2}$$

Shear

$$V = \frac{W}{2L^2}(L^2 - 4x^2)$$

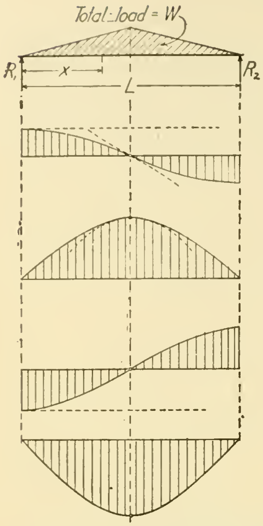
Moment

$$M = \frac{W}{6L^2}(3L^2x - 4x^3)$$

$M(\text{max.}) = \dfrac{WL}{6}$, at the center.

TABLE III, *continued*

9 (*Continued*)



Slope

$$EIv = -\frac{W}{12}\left(\frac{2x^4}{L^2} + \frac{5L^2}{8} - 3x^2\right)$$

$$EIv(\text{max.}) = \pm\frac{5WL^2}{96}, \text{ at the supports.}$$

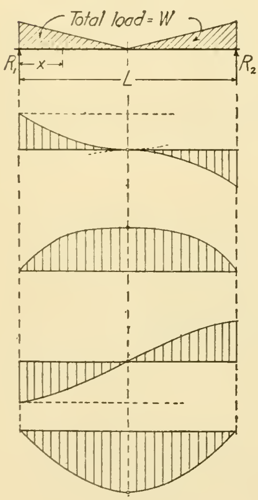
Deflection

$$EIy = -\frac{Wx^5}{30L^2} + \frac{Wx^3}{12} - \frac{5WL^2x}{96}$$

$$EIy(\text{max.}) = -\frac{WL^3}{60}, \text{ at the center.}$$

10

Note.—The loading conditions in this case are symmetrical. Thereore all equations are given for the left half of the beam only.



Loading

Total Load $= W$

Rate of Loading: at any point

$$= -\frac{2W}{L} + \frac{4Wx}{L^2}$$

$$\text{at the ends} = -\frac{2W}{L}$$

$$R_1 = R_2 = \frac{W}{2}$$

Shear

$$V = \frac{W}{2L^2}(L^2 - 4Lx + 4x^2)$$

Moment

$$M = \frac{W}{6L^2}(3L^2x - 6Lx^2 + 4x^3)$$

$$M(\text{max.}) = \frac{WL}{12}, \text{ at the center.}$$

Slope

$$EIv = -\frac{W}{96}\left(3L^2 - \frac{16x^4}{L^2} + \frac{32x^3}{L} - 24x^2\right)$$

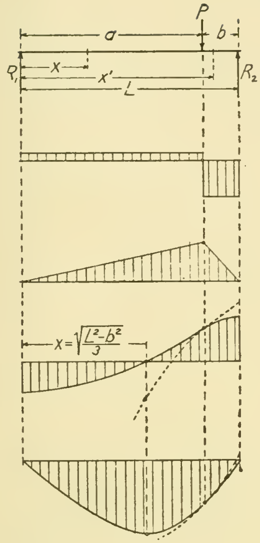
$$EIv(\text{max.}) = \pm\frac{WL^2}{32}, \text{ at the supports.}$$

Deflection

$$EIy = \frac{Wx^3}{12} - \frac{Wx^4}{12L} + \frac{Wx^5}{30L^2} - \frac{WL^2x}{32}$$

$$EIy(\text{max.}) = -\frac{3WL^3}{320}$$

TABLE III, *continued*



Load $= P$ 11

Shear

$$\text{I,}\ V = \frac{Pb}{L}$$
$$\text{II,}\ V' = -\frac{Pa}{L}$$

Moment

$$\text{I,}\ M = \frac{Pbx}{L}$$
$$\text{II,}\ M' = \frac{Pbx'}{L} - P(x' - a)$$
$$M(\text{max.}) = M'(\text{max.}) = \frac{Pba}{L}$$

Slope

$$\text{I,}\ EIv = \frac{Pb}{6L}[3x^2 - (L^2 - b^2)]$$
$$\text{II,}\ EIv' = \frac{Pb}{6L}\left[3x'^2 - \frac{3L}{b}(x' - a)^2 - (L^2 - b^2)\right]$$
$$^*EIv = 0 \text{ when } x = \sqrt{\frac{L^2 - b^2}{3}}$$

Deflection

$$\text{I,}\ EIy = \frac{Pb}{6L}[x^3 - (L^2 - b^2)x]$$
$$\text{II,}\ EIy' = \frac{Pb}{6L}\left[x'^3 - (L^2 - b^2)x' - \frac{L}{b}(x' - a)^3\right]$$
$$EIy(\text{max.}) = -\frac{Pb}{9\sqrt{3}L}(L^2 - b^2)^{3/2}, \text{ when } x = \sqrt{\frac{L^2 - b^2}{3}}$$

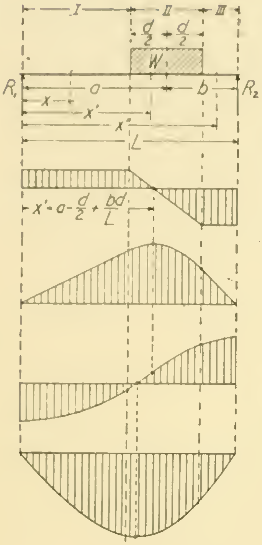
Note.—If the position of P is considered as variable;

M(max.) occurs when $a = \frac{L}{2}$ and is equal to $\frac{PL}{4}$

EIy(max.) occurs when $a = \frac{L}{2}$ and is equal to $-\frac{PL^3}{48}$

* In the above equations $a > b$.

TABLE III, *continued*



Loads 12

$$-\frac{W}{L}\text{ lbs. per ft.}\qquad R_1 = \frac{Wb}{L}\qquad R_2 = \frac{Wa}{L}$$

Shear

$$\text{I, } V = \frac{Wb}{L}$$

$$\text{II, } V' = \frac{Wb}{L} - \frac{W}{L}\left(x' - a + \frac{d}{2}\right)$$

$$\text{III, } V'' = \frac{Wb}{L} - W$$

Moment

$$\text{I, } M = \frac{Wb}{L}x$$

$$\text{II, } M' = \frac{Wb}{L}x' - \frac{W}{2L}\left(x' - a + \frac{d}{2}\right)^2$$

$$\text{III, } M'' = \frac{Wb}{L}x'' - W(x'' - a)$$

Slope

$$\text{I, } EIv = \frac{Wb}{24L}[12x^2 - 4(L^2 - b^2) + d^2]$$

$$\text{II, } EIv' = \frac{Wb}{24L}[12x'^2 - 4(L^2 - b^2) + d^2] - \frac{W}{6d}\left(x' - a + \frac{d}{2}\right)^3$$

$$\text{III, } EIv'' = \frac{Wb}{24L}[12x''^2 - 4(L^2 - b^2) + d^2] - \frac{W}{2}(x'' - a)^2 - \frac{Wd^2}{24}$$

Deflection

$$\text{I, } EIy = \frac{Wbx}{24L}[4x^2 - 4(L^2 - b^2) + d^2]$$

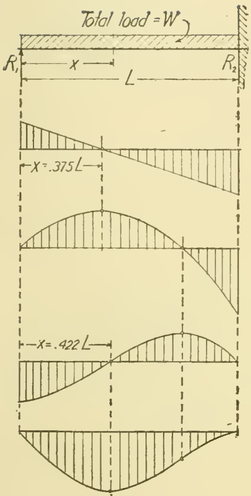
$$\text{II, } EIy' = \frac{Wbx'}{24L}[4x'^2 - 4(L^2 - b^2) + d^2] - \frac{W}{24d}\left(x' - a + \frac{d}{2}\right)^4$$

$$\text{III, } EIy'' = \frac{Wbx''}{24L}[4x''^2 - 4(L^2 - b^2) + d^2] - \frac{W}{6}(x'' - a)^3 - \frac{Wd^2}{24}(x'' - a)$$

Note.—For M'(max.)—Find the value of x' when $V' = 0$ and use this value of x' in moment equation II.

For EIy'(max.)—Find the value of x' when $EIv' = 0$ and use this value of x' in deflection equation II.

TABLE III, *continued*



Load 13

Rate of Loading $= -\frac{W}{L}$ lbs. per ft.

Total Load $= W$

$R_1 = \frac{3}{8}W$

Shear

$V = \frac{3}{8}W - \frac{Wx}{L}$

$V(\text{max.}) = -\frac{5}{8}W$, at restrained end.

$V = 0$, when $x = \frac{3}{8}L$

Moment

$M = \frac{3}{8}Wx - \frac{Wx^2}{2L}$

$M = 0$, when $x = \frac{3}{4}L$

$$M(\text{max.}) = \begin{cases} +\frac{9}{128}WL, \text{ when } x = \frac{3}{8}L \\ -\frac{1}{8}WL, \text{ when } x = L \end{cases}$$

Slope

$EIv = \frac{W}{48}(9x^2 - 8x^3 - L^2)$

$EIv = 0$, when $x = .422L$

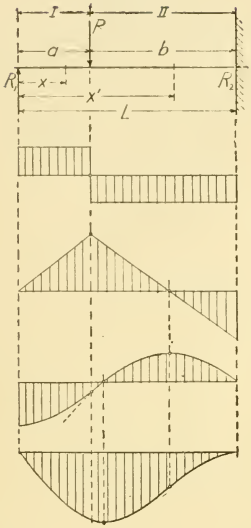
$EIv(\text{max.}) = -\frac{1}{48}WL^2$, when $x = 0$.

Deflection

$EIy = -\frac{W}{48}\left(\frac{2x^4}{L} + L^2x - 3x^3\right)$

$EIy(\text{max.}) = .0054WL^3$, when $x = .422L$.

TABLE III, *continued*



Load $= P$ **14**

$$R_1 = \frac{Pb^2}{2L^3}(2b + 3a)$$

Shear

I, $V = R_1$
II, $V' = R_1 - P$

Moment

I, $M = R_1 x$
$M(\text{max.}) = R_1 a$, under the load.
II, $M' = R_1 x' - P(x' - a)$

$$M'(\text{max.}) = \begin{cases} R_1 a, \text{ under the load.} \\ R_1 L - Pb, \text{ at the restrained end.} \end{cases}$$

$$M' = 0 \text{ when } x' = \frac{Pa}{P - R_1}$$

Slope

$$\text{I, } EIv = \frac{R_1 x^2}{2} + \frac{Pa^2}{2} - \frac{V'L^2}{2} - PaL$$

$$\text{II, } EIv' = \frac{V'x'^2}{2} + Pax' - \frac{V'L^2}{2} - PaL$$

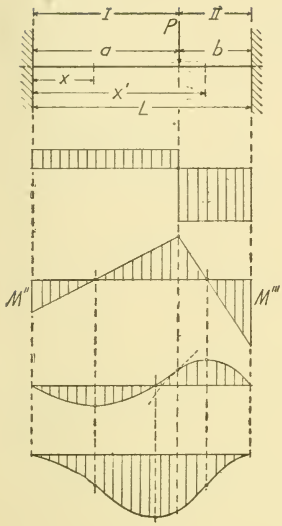
Deflection

$$\text{I, } EIy = \frac{R_1 x^3}{6} + \frac{Pa^2 x}{2} - \frac{V'L^2 x}{2} - PaLx$$

$$\text{II, } EIy' = \frac{V'x'^3}{6} + \frac{Pax'^2}{2} - \frac{V'L^2 x'}{2} - PaLx' + \frac{PaL^2}{2} + \frac{V'L^3}{3}.$$

Note.—If the position of P is considered as variable, the greatest possible value of M occurs when $a = .375L$ (about).
The maximum negative moment occurs at the restrained end, when $a = .577L$ (about).

TABLE III, *continued*



15

Load $= P$

Shear

I, $V = \frac{Pb^2}{L^3}(b + 3a)$

II, $V' = \frac{Pb^2}{L^3}(b + 3a) - P$

Moment

I, $M = M'' + Vx$

M'' (at left end) $= -\frac{Pab^2}{L^2}$

$M(\text{max. } +) = \frac{2Pa^2b^2}{L^3}$ (under the load).

$M = 0$, when $x = \frac{aL}{3a + b}$

II, $M' = \frac{Pb^2x'}{L^3}(b + 3a) - \frac{Pab^2}{L^2} - P(x' - a)$

M''' (at right end) $= -\frac{Pb}{L^2}(a^2 + a - ab)$

M' (max. +) = same as for M.

$M' = 0$, when $x = a + \frac{2b^2}{a + 3b}$

Slope

I, $EIv = \frac{Pb^2x}{2L^3}[x(3a + b) - 2aL]$

II, $EIv' = \frac{Pb^2x'}{2L^3}[x'(3a + b) - 2aL] - \frac{P}{2}(x' - a)^2$

Deflection

I, $EIy = \frac{Pb^2x^2}{6L^3}[x(3a + b) - 3aL]$

II, $EIy' = \frac{Pb^2x'^2}{6L^3}[x'(3a + b) - 3aL] - \frac{P}{6}(x' - a)^3$

EIy (max.) $= -\frac{2Pa^3b^2}{3(3a + b)^2}$, when $x = \frac{2aL}{3a + b}$

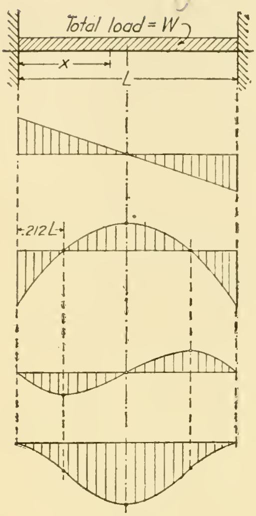
Note.—If the position of P is considered as variable;

The greatest possible + moment $= \frac{1}{8}PL$, when $a = \frac{L}{2}$

The greatest possible − moment $= \frac{4}{27}PL$, when $a = \begin{cases} \frac{1}{3}L \\ \frac{2}{3}L \end{cases}$

The greatest possible $EIy = \frac{1}{192}PL^3$, when $a = \frac{L}{2}$

TABLE III, *continued*



Load **16**

Rate of Loading $= -\dfrac{W}{L}$ lbs. per ft.

Total Load $= W$.

Shear

$$V = \frac{W}{2} - \frac{W}{L}x.$$

Moment

$$M = \frac{Wx}{2} - \frac{Wx^2}{2L} - \frac{WL}{12}$$

$$M(\text{max.}) = \begin{cases} +\dfrac{WL}{24}, \text{ at the center.} \\ -\dfrac{WL}{12}, \text{ at the ends.} \end{cases}$$

$$M = 0 \text{ when } x = \begin{cases} .212L \\ .788L \end{cases}$$

Slope

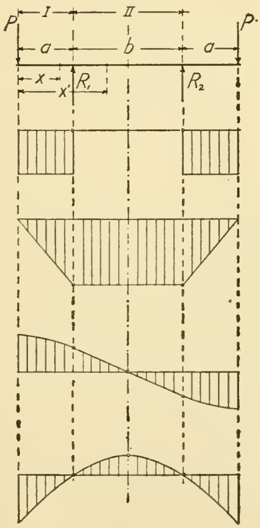
$$EIv = \frac{Wx^2}{4} - \frac{Wx^3}{6L} - \frac{WLx}{12}$$

$$EIv(\text{max.}) = \pm .00965WL^2, \text{ when } x = \begin{cases} .212L \\ .788L \end{cases}$$

Deflection

$$EIy = \frac{Wx^3}{12} - \frac{Wx^4}{24L} - \frac{WLx^2}{24}$$

$$EIy(\text{max.}) = -\frac{WL^3}{384}, \text{ when } x = \frac{L}{2}$$



Each Load $= P$ **17**

Shear

I, $V = -P$
II, $V' = 0$

Moment

I, $M = -Px$
$M(\text{max.}) = -Pa$
II, $M' = -Pa$

Slope

I, $EIv = \dfrac{P}{2}(a^2 + ab - x^2)$

II, $EIv' = Pa\left(a + \dfrac{b}{2} - x'\right)$

Deflection

I, $EIy = \dfrac{P}{6}[3abx + 3a^2x - x^3 - 3a^2b - 2a^3]$

$EIy(\text{max.}) = \dfrac{P}{6}[-3a^2b - 2a^3]$, at the end.

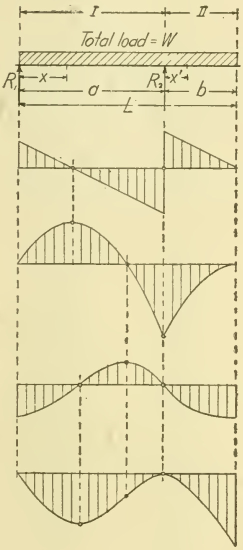
II, $EIy' = \dfrac{Pa}{2}[b(x' - a) - (x' - a)^2]$

$EIy'(\text{max.}) = \dfrac{Pab^2}{8}$, at the center.

Note.—The curves for the End Spans are symmetric.

TABLE III, *continued*

18



Load

Rate of Loading $= -\dfrac{W}{L}$ lbs. per ft.

Total Load $= W$.

$$R_1 = \frac{W}{2a}(a - b) \qquad R_2 = \frac{WL}{2a}$$

Shear

$$\text{I,}\quad V = R_1 - \frac{W}{L}x$$

$$V = 0, \text{ when } x = \frac{R_1 L}{W}$$

$$\text{II,}\quad V' = \frac{W}{L}(b - x')$$

Moment

$$\text{I,}\quad M = R_1 x - \frac{Wx^2}{2L}$$

$$M(\text{max.}) = \frac{R_1^2 L}{2W}, \text{ when } x = \frac{R_1 L}{W}$$

$$M = 0, \text{ when } x = \frac{2R_1 L}{W}$$

$$\text{II,}\quad M' = -\frac{W(b - x')^2}{2L}$$

$$M'(\text{max.}) = -\frac{Wb^2}{2L}, \text{ when } x' = 0$$

Slope

$$\text{I,}\quad EIv = \frac{1}{24}\left[4R_1(3x^2 - a^2) + \frac{W}{L}(a^3 - 4x^3)\right]$$

$$\text{II,}\quad EIv' = \frac{1}{24}\left\{8R_1 a^2 + \frac{W}{L}[4(b - x')^3 - 3a^3 - 4b^3]\right\}$$

Deflection

$$\text{I,}\quad EIy = -\frac{1}{24}\left[\frac{W}{L}(x^4 - a^3 x) - 4R_1(x^3 - a^2 x)\right]$$

$$\text{II,}\quad EIy' = -\frac{1}{24}\left[\frac{W}{L}(6b^2 x'^2 - 4bx'^3 + x'^4 + 3a^3 x') - 8R_1 a^2 x'\right]$$

TABLE IV

Coefficients of Friction

Note. The following table of coefficients of friction has been condensed from various sources. It is intended to give a definite basis for the solution of problems rather than accurate information on specific cases. So much depends on the condition of the surfaces as to finish, presence of moisture, etc., that no single value can be considered more than a rough average for a given case.

Earth on earth, dry	0.55
" " " , damp clay	1.00
" " " , wet	0.33
Masonry on dry earth	0.50
" " moist clay	0.35
" " masonry	0.65
" " timber	0.40
Metal on metal, dry	0.20
" " " , oiled	0.05
Metal on earth, dry	0.40
" " " , wet	0.30
" " masonry	0.35
" " timber	0.40
Stone on stone	0.58
Timber on timber	0.40
" " earth	0.40

TABLE V

Data Recorded During Tensile Tests on Steel and Cast Iron Made at the Watertown Arsenal

A Steel specimen. Diameter 0.505″. Sectional area 0.20 sq. in. Tested in tension.			*B* Cast iron specimen. Diameter 1.129″ Sectional area 1.00 sq. in. Tested in tension.	
Applied loads		Elongation per inch	Applied loads	Elongation per inch
Total	Per square inch		Per square inch	
200	1,000	0.	1,000	0.
1,000	5,000	0.00010	2,000	0.000050
2,000	10,000	.00040	3,000	.000100
4,000	20,000	.00065	4,000	.000150
6,000	30,000	.00095	5,000	.000205
7,000	35,000	.00110	6,000	.000275
8,000	40,000	.00125	7,000	.000345
8,400	42,000	.00130	8,000	.000405
8,600	43,000	.00135	9,000	.000480
8,800	44,000	.00140	10,000	.000550
9,000	45,000	.00145	11,000	.000630
9,200	46,000	.00150	12,000	.000715
9,400	47,000	.00155	13,000	.000800
9,600	48,000	.00160	14,000	.000900
9,800	49,000	.00170	15,000	.001000
10,000	50,000	.00220	16,000	.001105
10,200	51,000	.00415	17,000	.001210
10,400	52,000	.00560	18,000	.001400
10,800	54,000	.00780	19,000	.001500
11,200	56,000	.01000	20,000	.001660
11,600	58,000	.01190	21,000	.001830
12,000	60,000	.01425	23,000	.002315
12,400	62,000	.01635	25,000	.002945
17,360	86,800	.21000	28,020	....

INDEX